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APPARENTLY CHAOTIC ORBITS EMBEDDED IN CLOSED CURVES*

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Abstract. In [S. Bahar, *Chaos Solitons Fractals*, 5 (1995), pp. 1001–1006; 7 (1996), pp. 41–47], pictures were presented of the forward orbit $\{z_n\}_n$ in the plane, where z_n satisfies $z_{n+1} = z_n A_{n+1}$ for certain *affine* transformations A_n. These orbits (a) appear to lie on *closed curves*, and (b) assume forms reminiscent of strange attractors. We answer the questions raised in these references and prove that indeed in the aperiodic case the invariant orbits of this "iterated function system" lie on closed curves and that in some sense the maps act "chaotically" on the set of closed curves. A key role is being played by almost periodic functions. Furthermore, we provide an expression for the top Lyapunov exponent, that is, a sum in the periodic case and an integral in the aperiodic case. Our results have important implications in applications of computer visualization and imaging and shed some light on nonlinear dynamical systems.

Key words. almost periodic, attractor, iterated function system, Lyapunov exponent, nonlinear, orbit, self-affine

AMS subject classifications. Primary, 28A80; Secondary, 58F08, 58F12, 58F22

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1. Introduction.

Iterated function systems (IFS) have received tremendous attention in the literature over the past few years, in large part for their power to approximate and generate images [3]. A less well-known feature of IFS was investigated by Berger [4], who let one random (infinite) trajectory trace out an orbit on a smooth curve or wavelet, replacing the recursive, and thus, more cumbersome algorithm of successive corner cutting. In the same spirit, Massopust [9] studied a class of IFS whose attractors are C^1- and C^2-interpolating functions or C^n-interpolating surfaces. Most recently, Bahar [1, 2] presented some pictures of the orbital behavior of certain affine IFS. It was observed that these attractors exhibit complex behavior, as well as (a) appear to lie on *closed curves*, and (b) assume forms strikingly similar to strange attractors, especially when the IFS is stochastic.

In this paper, we consider the forward orbit $\{z_n\}_n$, where z_n satisfies $z_{n+1} = z_n A_n$ for some affine transformations A_n, and we answer the two questions raised (but not answered) in [1, 2], namely, (1) do these orbits, left invariant under affine transformations, live on closed curves, and (2) what can be said about the shape and geometry of these orbits; in particular, do they exhibit *chaotic* behavior? The latter conjecture will be confirmed in the sense described below (see next paragraph). Moreover, an easily checkable condition will be provided that ensures stability of the limiting orbit (as opposed to divergence to infinity if the system is repelling). The established results will have several interesting implications in applications.

Indeed, we will prove that, in the *aperiodic* case, the orbit, extracted from the invariant set of the IFS, lies on a closed curve that we specify, as the number of it-

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erations tends to infinity. This closed curve may consist of numerous loops so that the curve well approximates a surface. The periodic case is degenerate and produces a finite number of points (compare to the irrational and rational rotations or toral automorphisms that are well studied in the literature). Nevertheless, only the exceptional case relative to Lebesgue measure, the periodic case, is accessible to these orbits when generated by computer. Observe that the IFS has some striking features. Let us mention one or two. Slightly changing one of the two scale parameters, while keeping the other one fixed (for instance, from $b_2 = 10000$ to $b_2 = 10005$, while fixing $a_2 = 10000$, as illustrated in Figures 2 and 3), may cause an explosion in the number of loops along with a severe change in the shape of the curve. These discontinuities in the space of closed curves that arise by perturbing the scale parameters within the periodic regime, within the aperiodic regime, or by switching between periodic and aperiodic regimes provide the chaotic nature of the system relative to the scale parameters (for a definition of "chaos," see [7, Chapter 13, p. 171]). Moreover, the fact that the time-dependent maps are *nonlinear* in time and also "couple" the x- and y-coordinates of the orbit equips our dynamical system with the chaotic appearance that is typical in nonlinear dynamical systems and strange attractors. Of course, this applies only to the aperiodic case. Yet the better the aperiodic case is "approximated," the closer the system will get to chaos. The dependencies on the parameters are discontinuous. There are many interesting consequences for applications. Next, we shall discuss a few and we hope to spur interest for others.

Due to its simplicity, the proposed system should be helpful to generate a surface, given the boundary of the surface, or to generate the boundary of a region, given the region. The former is useful in computer visualization, the latter in imaging—more precisely, to encode and decode images. For instance, in the storing process of medical images of the heart or the brain, certain boundary lines can contain the most crucial information and, given the limited storage space for these routine medical images, it is not necessary to encode the entire image. Even more, by doing so the sharpness of the boundary lines may be lost after decoding because they are lower-dimensional objects. Since, as we will see, the system that we propose is relatively simple, tuning the parameters to obtain the desired curve is not difficult.

Moreover, this class of IFS promises to shed some light on the less understood properties of attractors of nonlinear systems, for which it is difficult to rigorously calculate Lyapunov exponents along with other important quantities. The emerging nonlinear behavior of this model offers us the opportunity to investigate the transition to the chaotic regime.

Iterating in "random environment" and selecting the affinities randomly at each step reinforces the peculiar and complex appearance of the orbits since several orbits generated by the deterministic scheme are superimposed, although the behavior of random orbits is similar to the one explained in this paper. Here we shall omit the mathematical details and present only a few figures; we will more thoroughly treat that situation in a separate paper.

1.1. Definition and background: Forward orbit and IFS. We begin with describing the particulars of the system at hand. Let $s : \mathbf{R} \to \mathbf{R}$ and $c : \mathbf{R} \to \mathbf{R}$ be two continuous periodic functions of periods p_1 and p_2. For every integer $n > 0$ and any real numbers a_1, a_2, b_1, and b_2, define

$$\begin{aligned} m_n &= a_1\, s(n/a_2), \\ v_n &= b_1\, c(n/b_2), \end{aligned} \tag{1.1}$$

and for any real numbers d_1 and d_2, define the matrix

$$T_n = \begin{pmatrix} m_n & d_1 m_n \\ v_n & d_2 v_n \end{pmatrix}.$$

For any real numbers α and β and some initial point $z_0' = (x_0, y_0) \in \mathbf{R}^2$, consider the points $\{z_n' = (x_n, y_n)\}$ given by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = T_{n+1} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where z' denotes the transpose of the vector z. We will be interested in the limiting forward orbit $\{z_n\}_{n \gg 1}$ in the contractive case. Specifics on a contraction condition will be presented shortly. For n large enough, the sequence $\{z_n\}_n$ will be independent of the initial point (x_0, y_0). A simple but illuminating example for m_n and v_n that the reader may keep in mind during the subsequent discussion is $m_n = a_1 \sin(n/a_2)$ and $v_n = b_1 \cos(n/b_2)$. In fact, all figures use $s(x) = \sin(x)$ and $c(x) = \cos(x)$.

Our first observation is that there are two completely different cases to distinguish, namely, (1) the *periodic* case, that is, when there exists some positive integer P such that $T_{i+P} = T_i$ for every integer $i > 0$ (take the smallest such integer P, called the period); and (2) the *aperiodic* case. For the purpose of applications, the latter case is the interesting one. In the aperiodic context, $\{m_n\}_{n>0}$ and $\{v_n\}_{n>0}$, viewed as functions embedded in the reals, are *almost periodic* functions (see Definition 1.1), and thus, the affine transformations

$$A_n = T_n + (\alpha, \beta)' \tag{1.2}$$

are almost periodic as well. Next, because at each step n there is exactly one prescribed transformation A_n to be applied, note that the limiting orbit of $\{z_n\}_{n \geq 1}$ is a subset of the (unique) invariant set of an affine iterated function system with a *finite* number of affine transformations in the periodic case and an *infinite* number of affine transformations in the aperiodic case. Let us first regard the orbit in the periodic case. Suppose that a_2 and b_2 are chosen such that there exists some positive integer P such that $(m_{n+P}, v_{n+P}) = (m_n, v_n)$ for all integers n. This happens if and only if $a_2 = P/(p_1 k_1)$ and $b_2 = P/(p_2 k_2)$ for some integers k_1 and k_2. This implies that $A_{n+P} = A_n$ for each integer $n \geq 0$; thus, the number of affine transformations $\{A_1, A_2, \ldots, A_P\}$ is finite. Hence, it is an easy instance of the contraction principle in the theory of IFS [8] that, if some contraction condition is satisfied, each trajectory $A_n \cdot A_{n-1} \cdot \ldots \cdot A_1 z_0$ will repeat itself forever. Indeed, each limit

$$\lim_{n \to \infty} A_{i_1} A_{i_2} \cdot \ldots \cdot A_{i_n} z_0$$

is independent of the point z_0. But since $A_{i_1} A_{i_2} \cdot \ldots$ is a periodic semi-infinite sequence, there are *exactly* P distinct such limits, one for each possible starting map A_{i_1} (equivalently, for each beginning segment $A_{i_1} A_{i_2} \cdot \ldots \cdot A_{i_P}$). This implies that, since $z_n = A_n A_{n-1} \cdot \ldots \cdot A_2 A_1 z_0$, there are no more than P distinct limit points z_i^*, for $1 \leq i \leq P$. These P limit points must coincide with the fixed points of $A_1, A_2, \ldots, A_P$, whose coordinates can be easily calculated. In light of the almost-periodicity of the linear transformations, the aperiodic case derives from the periodic case.

A sufficient condition that guarantees contractivity of the IFS described above is $\|T_i\| < 1$ for each integer $i > 0$. We will prove that this condition may be relaxed

(Theorem 1.3). In fact, it suffices to require that the (pointwise) top Lyapunov exponent of the system be negative in order to establish stability of the orbit $\{z_n\}_n$ for each sufficiently large n. Although geometrically the aperiodic case is dramatically different from the periodic case (the reader may try to plot some trajectories), the situation concerning contractivity and existence of a limit set is analogous to the one explained, regardless of the infinity of the number of underlying maps. Indeed, an approximate period (see Definition 1.2) may be chosen such that the periodic case can be mimicked.

Figures. Prior to stating the main results, we make some general observations and glimpse at the figures. To build up intuition and to help understand the main ideas, we return to the example where $m_n = a_1 \sin(n/a_2)$ and $v_n = b_1 \cos(n/b_2)$ for each $n > 0$. Thus, $p_1 = p_2 = 2\pi$. In Figures 2–5, $d_1 = d_2 = 1$. Each of the parameters has its impact on the system. The most vigorous part is being played by the parameters a_2 and b_2. While a_1 and b_1 tune the "amplitude" of the image, the parameters a_2 and b_2 determine the "wavelength." The orbits behave discontinuously in a_2 and b_2. The invariant set will be shaped very differently depending on whether b_2/a_2 is rational (periodic case) or irrational (aperiodic case), and depending on whether b_2/a_2 equals some integer L, say, or equals $L + \delta$ for some small real $\delta \neq 0$. Every irrational δ will lead to an orbit that is dense in a closed curve. Examples with $b_2/a_2 = 1$ and $b_2/a_2 = 1.0005$, respectively, are depicted in Figure 2 and Figure 3, respectively. For a countable number of values of (a_2, b_2) the orbits behave discontinuously; in other words, two arbitrarily nearby values of a_2, say, will produce two closed curves differing significantly in shape and in density of the points.

Examples of invariant orbits under random affine transformations are shown in Figures 4 and 5. At each step, according to a Bernoulli(0.5) random variable, one of two possible affine transformations is picked, with one having the opposite order of the rows to the other relative to sin and cos. The parameter values are $a_2 = b_2 = 0.2$ in Figure 4 and $a_2 = 0.2$ and $b_2 = 0.2001$ in Figure 5. In both figures, $a_1 = 0.3$, $b_1 = 0.45$, $\alpha_1 = \alpha_2 = 0.1$, and $\beta_1 = \beta_2 = 0.2$. In fact, the symmetry in the image results from the symmetry between the two transformations.

Finally, Figure 1 presents a random orbit, as do Figures 4 and 5, with the only difference being the multipliers of the row elements, that is, d_i are not all equal 1.

1.2. Almost periodic functions and matrices. A key ingredient of our proofs is the notion of an almost periodic function. A good reference is [5]. We present the definition.

DEFINITION 1.1. *Fix some $\varepsilon > 0$. A real number $\tau = \tau_\varepsilon$ is a* translation number *of the function $f : \mathbf{R} \to \mathbf{R}^n$ relative to ε if*

$$\|f(x + \tau) - f(x)\| \leq \varepsilon$$

for all $x \in \mathbf{R}$. A continuous function on the reals is called almost periodic *if, for any $\varepsilon > 0$, there is a length $L = L(\varepsilon)$ such that each interval in $\mathbf{R}$ of length L contains at least one translation number τ_ε.*

Clearly, every continuous periodic function is almost periodic (with the translation numbers forming an arithmetic sequence), and each almost periodic function over $\mathbf{Z}$ is almost periodic over $\mathbf{R}$. In the nondegenerate (aperiodic) case, an elementary exercise shows that $\{(m_k, v_k)\}_{k\geq 1}$ is almost periodic. Observe that any bounded continuous function of almost periodic functions is an almost periodic function. In particular, finite products or sums of almost periodic functions are almost periodic functions,

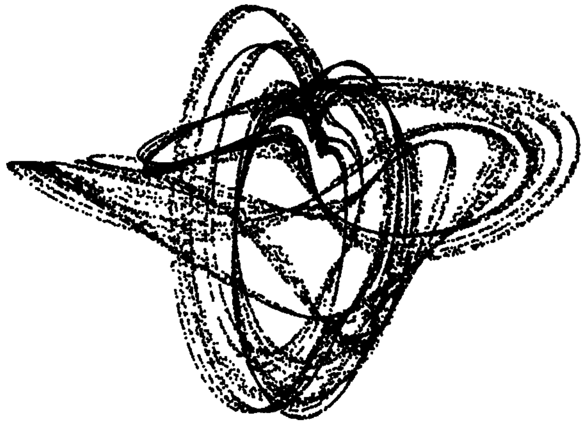

FIG. 1. *An affine random IFS attractor. Here, we have two transformations, each assigned equal probability. For both transformations,* $a_1 = 0.3$, $b_1 = 0.45$, $a_2 = 0.42$, $b_2 = 0.42$, *and* $\alpha_1 = \alpha_2 = 0.1$, $\beta_1 = \beta_2 = 0.2$. *In the first transformation, however, the first row of* T_n *is multiplied by* (0.7, 0.9) *and the second row by* (1.0, 1.0)*, while in the second transformation the first row of* T_n *is multiplied by* (1.0, 2.5) *and the second row by* (1.0, −1.0).

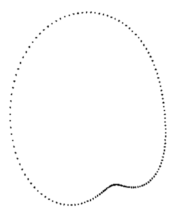

FIG. 2. *An orbit where* $L = b_2/a_2 = 1$ *is an integer. We have generated an image via an IFS with coefficients* m_n *and* v_n *defined as in* (1.1)*. The parameters are* $a_1 = 0.3$, $b_1 = 0.45$, $a_2 = 0.2$, $b_2 = 0.2$, $\alpha = 0.1$, $\beta = 0.2$. *The image is plotted in the xy plane after discarding the first* 70 *points.*

too. Furthermore, solutions of certain systems of linear equations whose coefficients are almost periodic functions are almost periodic functions. Also, note that the above notion is different from the notion of *quasi periodicity.*

We will say that the "least" positive translation number τ_ε of some almost periodic function f is an approximate period.

DEFINITION 1.2. *For any fixed* $\varepsilon > 0$, *define the* approximate period $p = p_\varepsilon$ *of* f *to be the smallest positive real such that* $\|f(x + p_\varepsilon) - f(x)\| \leq \varepsilon$ *for all* $x \in \mathbf{R}$.

If $p_1 = p_2$, then $p_\varepsilon = p_1$ for all $\varepsilon > 0$.

1.3. Main results: Closed curves and Lyapunov exponents. For any real number d, define

$$f(x) = a_1 s(x/a_2) + d b_1 c(x/b_2) \tag{1.3}$$

and let Z_f denote the *set of zeros* of the function f. Now we are ready to state the principal results of this paper.

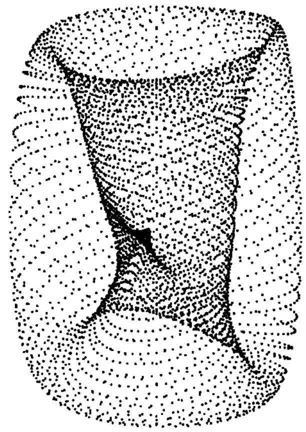

FIG. 3. *An attractor with $L = b_2/a_2 = 1.0005$. Parameters are identical to those used to generate Figure 2, except that $b_2 = 0.2001$.*

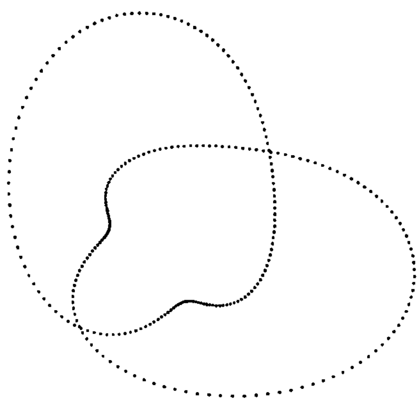

FIG. 4. *An orbit for a random IFS. Parameters are $a_1 = 0.3$, $b_1 = 0.45$, $a_2 = 0.2$, $b_2 = 0.2$, $\alpha_1 = \alpha_2 = 0.1$, $\beta_1 = \beta_2 = 0.2$.*

THEOREM 1.3. *Assume that $d_1 = d_2 = d$ and that $m_k + dv_k \neq 0$ for every integer $k > 0$.*

(I) Aperiodic case. *Assume that the parameters a_1, a_2, b_1, and b_2 are chosen such that $f'(x) \neq 0$ for $x \in Z_f$. If the limit*

$$\lim_{n\to\infty} \frac{\ln(m_n^2 + v_n^2)}{n} \tag{1.4}$$

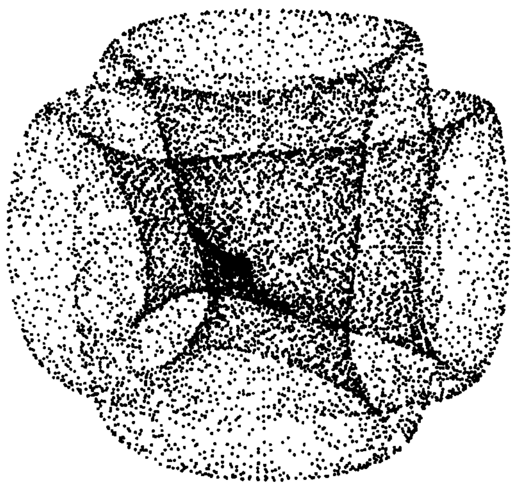

FIG. 5. *An attractor for a random IFS with multiple loops. Parameters are* $a_1 = 0.3$, $b_1 = 0.45$, $a_2 = 0.2$, $b_2 = 0.2001$, $\alpha_1 = \alpha_2 = 0.1$, $\beta_1 = \beta_2 = 0.2$.

exists, then there is some finite constant K_f *such that, for every* $\varepsilon > 0$, *the top Lyapunov exponent* λ_1 *of the orbit* $\{(x_n, y_n)\}_{n>0}$ *satisfies* $\lambda_1 \geq K_f$ *and*

$$\left| \lambda_1 - \frac{1}{p_\varepsilon} \int_0^{p_\varepsilon} \ln |a_1 s(x/a_2) + d b_1 c(x/b_2)| \, dx \right| \leq \varepsilon. \tag{1.5}$$

In the special case $p_1 = p_2$, *we have* $\lambda_1 = p_1^{-1} \int_0^{p_1} \ln |f(x)| dx$. *For fixed* a_1 *and* b_1, *and for Lebesgue almost every choice of* a_2 *and* b_2, *the limit in* (1.4) *exists and* f' *is nonzero at the zeros of* f.

(II) Periodic case. *Assume that* P *is the period of* f. *Then the top Lyapunov exponent* λ_1 *of the orbit* $\{(x_n, y_n)\}_{n>0}$ *is given by*

$$\lambda_1 = \frac{1}{P} \sum_{j=1}^{P} \ln |m_j + d v_j|. \tag{1.6}$$

(For the definition of the Lyapunov exponents, see, e.g., [7, p. 191] or any standard reference on dynamical systems.) There are analogues of the closed expression for λ_1 in the general case $d_1 \neq d_2$, yet they are more complicated. Thus, we prefer to omit the details. Observe that the second Lyapunov exponent $\lambda_2 = -\infty$ if $d_1 = d_2$.

We notice that the limit $\lim_{n\to\infty} \ln(m_n^2 + v_n^2)/n$ does not always exist in the aperiodic case, as the following counterexample illustrates.

Counterexample. Let $m_k = a_1 \sin(k/a_2)$ and $v_k = b_1 \cos(k/b_2)$ for every integer k. Now we give some special choice of a_2 and b_2. Indeed, let the binary expansion of $1/(\pi a_2)$ be as follows:

$$\frac{1}{\pi a_2} = 0.1001000010\ldots010\ldots010\ldots,$$

where the number of zeros in the kth block of consecutive zeros to the right of the decimal point equals w_k with $w_{k+1} = 2^{w_k}$ and $w_1 = 2$ (and set $w_0 = 0$). Let the binary expansion of $1/b_2$ be

$$\frac{1}{b_2} = 0.d_1 \ldots d_2 \ldots d_3 \ldots d_4 \ldots,$$

where d_k indicates the beginning of the kth block of length w_k+1, each block consisting of the first binary digits of $\pi/2$, and w_k is given as before. Now, multiplying $1/a_2$ and $1/b_2$ by $n_k = 2^{u_k}$, where $u_k = k + \sum_{i=1}^{k} w_i$ for every integer $k \geq 0$, leads to a subsequence $\{n_k\}_{k\geq 1}$ such that $\lim_{k\to\infty} \ln(m_{n_k}^2 + v_{n_k}^2)/n_k = -\infty$.

For the next result, we do not assume that $d_1 = d_2$.

THEOREM 1.4. *Assume $d_1, d_2 \in \mathbf{R}$ and that the top Lyapunov exponent $\lambda_1 < 0$ (possibly, $-\infty$), where λ_1 is described in Theorem 1.3 for $d_1 = d_2 = d$.*

(I) Periodic case. *Assume that P is the period of f. Then for Lebesgue almost all choices of a_2 and b_2, and for sufficiently large $n = r(\text{mod } P)$, the orbit $\{(x_n, y_n)\}_{n>0}$ satisfies*

$$x_n = m_r\delta_5(r) + \alpha, \qquad (1.7)$$
$$y_n = v_r\delta_6(r) + \beta,$$

where the coordinates $\{\delta_i(r)\}_{0\leq r\leq P-1}$, $i = 5, 6$, are the solutions of the recursion in (2.7).

(II) Aperiodic case. *For sufficiently large n, the shifted orbit $\{(x_n - \alpha, y_n - \beta)\}_{n>0}$ is embedded in the closed curve*

$$(a_1 s(x/a_2)D_5(x), b_1 c(x/b_2)D_6(x))_{x\in\mathbf{R}}, \qquad (1.8)$$

where the distortion functions $D_5(x)$ and $D_6(x)$ are the continuous extensions over the reals of the sequences $\{\delta_5(r)\}_{0\leq r\leq P-1}$ and $\{\delta_6(r)\}_{0\leq r\leq P-1}$, respectively, with $P = p_\varepsilon$ for suitably small $\varepsilon > 0$.

The fact that the curve described in this result is *closed* is due to the almost periodicity of the underlying functions in the aperiodic case. Observe that the functions D_5 and D_6 can be calculated with any desired degree of accuracy by solving system (2.7) for an approximate period (instead of the period in the periodic case). Clearly, $d_1 = d_2$ implies $D_5 \equiv D_6$ or $\delta_5 \equiv \delta_6$.

Remarks.

(1) The above-mentioned results generalize to the higher-dimensional (> 2) Euclidean space without complicating the discussion. Furthermore, similar results can be proved if f is a linear combination of almost periodic functions.

(2) The method adopted shows that, for fixed integer $k \geq 1$, the *return map* $\{(x_n, x_{n+k})\}_{n\geq 1}$ of the first coordinate enjoys the very same properties as the orbit $\{z_n\}_{n\geq 1}$; thus in the aperiodic case, its values fall on a closed curve. Of course, this holds for the second coordinate y_n as well.

(3) Selecting the affine transformations randomly may reinforce the complexity of the orbit, although the behavior of the orbits under random transformations are not much different. In fact, the system considered in "random environment" leads to a superposition of a good number of such orbits that eventually are traced out with probability one (see Figures 1 and 5 for a random orbit).

The structure of the rest of the paper is as follows. Section 2 derives the coordinate functions of the orbits and proves Theorem 1.4. Section 3 discusses the top Lyapunov exponent of the system, presents some estimates to justify the integral expression for the Lyapunov exponent in the aperiodic case, and proves Theorem 1.3.

2. Limiting orbit. We will carry out the algebra on which the asymptotics will rest for the periodic case and derive the almost periodic case from the periodic context by means of a continuity argument.

2.1. Composition law. First, we examine the composition law and fix more notation. Recall that we consider the orbit $\{z_n,\, n \in \mathbf{N}\}$ in $\mathbf{R}^2$ under affine transformations, described by $z_n = T_n z_{n-1} + \gamma$, where $\gamma' = (\alpha, \beta)$ for some real numbers α, β and the matrices T_n introduced in section 1. Let $M_2(\mathbf{R})$ be the set of 2×2 matrices that have the first row elements proportional to m_n and the second row elements proportional to v_n for some integer n and define the product space $H = M_2(\mathbf{R}) \times \mathbf{R}^2$. Then H is closed under a certain *composition law* that we describe next. Let $E, F \in M_2(\mathbf{R})$ be given by

$$E = \begin{pmatrix} e_1 m_j & e_2 m_j \\ e_3 v_j & e_4 v_j \end{pmatrix}, \qquad F = \begin{pmatrix} f_1 m_{j-1} & f_2 m_{j-1} \\ f_3 v_{j-1} & f_4 v_{j-1} \end{pmatrix}$$

for $e_i, f_i \in \mathbf{R}$, $i = 1, 2, \ldots, 4$ and some integer $j > 1$, and let $\gamma_j' = (\alpha_j, \beta_j)$ for some real numbers α_j, β_j. Thus, $(E, \gamma_j'), (F, \gamma_{j-1}') \in H$ and $((E, \gamma_j'), (F, \gamma_{j-1}')) \in H$, with the composition law on H being defined by

$$((E, \gamma_j), (F, \gamma_{j-1})) = \left(\begin{pmatrix} m_j(e_1 f_1 m_{j-1} + e_2 f_3 v_{j-1}) & m_j(e_1 f_2 m_{j-1} + e_2 f_4 v_{j-1}) \\ v_j(e_3 f_1 m_{j-1} + e_4 f_3 v_{j-1}) & v_j(e_3 f_2 m_{j-1} + e_4 f_4 v_{j-1}) \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} m_j(\alpha_{j-1} + d_1 \beta_{j-1}) + \alpha_j \\ v_j(\alpha_{j-1} + d_2 \beta_{j-1}) + \beta_j \end{pmatrix} \right).$$

Observe that H is *not* a semigroup. Applying this to $(T_j, \gamma') \in H$ yields

$$(T_n, \gamma) \cdot (T_{n-1}, \gamma) \cdots (T_1, \gamma) = (M_n, k_n), \tag{2.1}$$

where M_n is the product $M_n = T_n T_{n-1} \cdots T_2 T_1$ and k_n accounts for the repeated shifts by $\gamma' = (\alpha, \beta)$. Hence,

$$z_n = M_n z_0 + k_n \tag{2.2}$$

for every initial point $z_0' = (x_0, y_0)$.

2.2. The attractors are embedded in closed curves. Next we discuss the behavior of the sequences $\{M_n\}_{n \geq 1}$ and $\{k_n\}_{n \geq 1}$. In the periodic case, care is required to ensure invertibility of the system below. Thus in the next result, there are some anomalies that have zero Lebesgue measure.

LEMMA 2.1. *For every integer $n > 1$, we have*

$$M_n = \begin{pmatrix} m_n\, \delta_1(n) & m_n\, \delta_2(n) \\ v_n\, \delta_3(n) & v_n\, \delta_4(n) \end{pmatrix},$$

where $\delta_i(n)$ satisfy the recursion

$$\begin{aligned} \delta_1(n) &= m_{n-1}\, \delta_1(n-1) + v_{n-1}\, d_1\, \delta_3(n-1), \\ \delta_2(n) &= m_{n-1}\, \delta_2(n-1) + v_{n-1}\, d_1\, \delta_4(n-1), \\ \delta_3(n) &= m_{n-1}\, \delta_1(n-1) + v_{n-1}\, d_2\, \delta_3(n-1), \\ \delta_4(n) &= m_{n-1}\, \delta_2(n-1) + v_{n-1}\, d_2\, \delta_4(n-1), \end{aligned} \tag{2.3}$$

and $\delta_1(1) = 1$, $\delta_2(1) = d_1$, $\delta_3(1) = 1$, *and* $\delta_4(1) = d_2$.

Moreover, assume that each T_i *has norm less than* 1. *Then in the aperiodic case, as* $n \to \infty$, *the matrices* M_n *converge to the zero matrix; that is, for every* $i = 1, 2, \ldots, 4$, *the numbers* $\delta_i(n) \to 0$ *as* $n \to \infty$. *In the periodic case (with period* P*), if, for every* $n > 1$, *we define the four discrete functions* g_1, g_2, h_1, *and* h_2 *by the recursion scheme*

$$
\begin{aligned}
g_1(k) &= m_{n-k}\,[g_1(k-1) + g_2(k-1)],\\
g_2(k) &= v_{n-k}\,[d_1 g_1(k-1) + d_2 g_2(k-1)],\\
h_1(k) &= m_{n-k}\,[h_1(k-1) + h_2(k-1)],\\
h_2(k) &= v_{n-k}\,[d_1 h_1(k-1) + d_2 h_2(k-1)]
\end{aligned} \tag{2.4}
$$

for $k = 2, 3, \ldots, n-1$ *and* $g_1(1) = h_1(1) = m_{n-1}$, $g_2(1) = d_1 v_{n-1}$, *and* $h_2(1) = d_2 v_{n-1}$, *and if we assume that the following algebraic conditions are satisfied:*

$$
\begin{aligned}
(1 - h_2(P))\,(1 - g_1(P)) &\neq h_1(P) g_2(P),\\
(1 - d_2 h_2(P))\,(1 - d_1 g_1(P)) &\neq d_1 d_2 h_1(P) g_2(P),
\end{aligned} \tag{2.5}
$$

then as $n \to \infty$, *the matrices* M_n *converge to the zero matrix.*

Proof. The first claim (2.3) is an elementary exercise that rests on induction over n. We omit the details.

To prove the second statement about the limiting behavior of the $\delta_i(n)$, we first address the *periodic* case, the period being denoted by P. It is sufficient to present the argument for the set of functions δ_1 and δ_3 since the argument runs in parallel for the functions δ_2 and δ_4.

The functions δ_1 and δ_3 follow the recursion

$$
\begin{aligned}
\delta_1(n) &= g_1(k)\,\delta_1(n-k) + g_2(k)\,\delta_3(n-k),\\
\delta_3(n) &= h_1(k)\,\delta_1(n-k) + h_2(k)\,\delta_3(n-k)
\end{aligned}
$$

for all integers $n > 1$ and $k > 0$, in particular for $k = P$, where g_1, g_2, h_1, and h_2 follow the recursion given in (2.4). Since $(\delta_1(n), \delta_3(n))$ satisfy the same recursion as $(\delta_1(n-P), \delta_3(n-P))$, it follows that $\delta_1(n) = \delta_1(n-P)$ and $\delta_3(n) = \delta_3(n-P)$. Therefore, the limits of the sequences $\delta_1(n)$ and $\delta_3(n)$, guaranteed by the assumption that each T_i has norm less than 1, is periodic and has the form $(\delta_{i1}, \delta_{i2}, \ldots, \delta_{iP})$ for $i = 1, 3$. Let $r = n \pmod P$. Then $(\delta_{1r}, \delta_{3r})$ satisfies

$$
\begin{aligned}
\delta_{1r} &= g_1(P)\,\delta_{1r} + g_2(P)\,\delta_{3r},\\
\delta_{3r} &= h_1(P)\,\delta_{1r} + h_2(P)\,\delta_{3r}.
\end{aligned} \tag{2.6}
$$

From the first equation in (2.6) we find that $\delta_{1r}(1 - g_1) = g_2 \delta_{3r}$, and from the second equation, $\delta_{3r}(1 - h_2) = h_1 \delta_{1r}$, where we have written $g_j = g_j(P)$ and $h_j = h_j(P)$ for $j = 1, 2$. Hence, we obtain

$$
\delta_{3r}(1 - h_2)(1 - g_1) = h_1\, g_2\, \delta_{3r},
$$

an equation whose validity is precluded by conditions (2.5), except for $\delta_{3r} = 0$. Therefore, we conclude that $\delta_{1r} = \delta_{3r} = 0$ for every $r = 1, 2, \ldots, P$.

Now, in the *aperiodic case,* it remains to be observed that for every $\varepsilon > 0$ there is a time $p = p_\varepsilon$ such that (m_{n+p}, v_{n+p}) is within ε-distance to (m_n, v_n), measured, say, in Euclidean distance. Hence, we find four linear equations that are arbitrarily close to the equations as given in the periodic case above. Therefore, any solutions δ_{ik} must be arbitrarily close to the solutions in the periodic case and thus to zero. In fact, the solutions δ_{ik} are almost periodic functions. We omit the ε-arguments behind our words "arbitrarily close." In this case, the algebraic conditions can be ignored, as one can always find some approximate period $p_\varepsilon = p$ such that the algebraic conditions are satisfied with $P = p_\varepsilon$. This finishes our proof. $\square$

LEMMA 2.2. *For every integer $n > 1$, we have*

$$k_n = \begin{pmatrix} m_n \, \delta_5(n) + \alpha \\ v_n \, \delta_6(n) + \beta \end{pmatrix},$$

where $\delta_5(n)$ and $\delta_6(n)$ satisfy the recursion

$$\begin{aligned} \delta_5(n) &= m_{n-1} \, \delta_5(n-1) + v_{n-1} \, d_1 \, \delta_6(n-1) + \alpha + d_1 \, \beta, \\ \delta_6(n) &= m_{n-1} \, \delta_5(n-1) + v_{n-1} \, d_2 \, \delta_6(n-1) + \alpha + d_2 \, \beta, \end{aligned} \tag{2.7}$$

and $\delta_5(1) = \delta_6(1) = 0$.

Proof. This is an immediate consequence of an induction argument over n. $\square$

This finishes the proof of Theorem 1.4. System (2.7) consists of linear equations in $\delta_5(\cdot)$ and $\delta_6(\cdot)$ and thus is straightforward to solve.

3. Lyapunov exponents when $d_1 = d_2$. The Lyapunov exponents measure the exponential decay of the dependence of the orbit on the initial conditions (for definitions and more see [6] or [7, p. 191]) and are useful to characterize the stability properties of the orbits. However, it is well known that, for nonlinear systems, it is generally difficult to obtain closed expressions for Lyapunov exponents. In IFS, such expressions are available. We shall be interested in the *top* Lyapunov exponent. To facilitate the presentation, we may assume that $d_1 = d_2 = d$. The developed results have analogues in the general case $d_1 \neq d_2$, with the algebra being more clumsy and the Lyapunov exponent being a pointwise Lyapunov exponent that takes different values at different points. For $d_1 \neq d_2$, the top Lyapunov exponent is a function of some expressions gotten as solutions of a recursion scheme analogous to the ones encountered in section 2.2. Moreover, note that the top Lyapunov exponent will provide a weaker contraction criterion than the requirement $\|T_i\| < 1$ for each integer $i > 0$.

It will turn out that, if $d_1 = d_2$, then the matrices T_n are singular; thus, one of the singular values equals zero and, equivalently, one of the two Lyapunov exponents is negative infinite. If the largest Lyapunov exponent of the system is finite, then it can be expressed as a finite sum in the periodic case and approximated by an integral in the aperiodic case (the integral is exact if $p_1 = p_2$). To justify the integral expression, we shall argue that the integral is finite, which in fact requires a few delicate estimates involving almost periodic functions and a "transversality condition" to be enjoyed by the function $f(x) = a_1 s(x/a_2) + d b_1 c(x/b_2)$ at its zeros. Since f has at most finitely many zeros on each interval of finite length, the minimal slope at the set of zeros is well defined. Recall that Z_f denotes the zero set of f and define

$$\rho = \rho_{f,\varepsilon} = \min_{x \in Z_f \cap [0, p_\varepsilon]} |f'(x)| \tag{3.1}$$

if $Z_f \cap [0, p_\varepsilon]$ is nonempty, and, $\rho = 1$, say, if $Z_f \cap [0, p_\varepsilon]$ is empty. For all that follows, we agree on the following two standard assumptions:

(A) $\rho > 0$, ("transversality condition").

(B) $m_j + dv_j \neq 0$ for every integer j.

Both assumptions can be shown to hold for Lebesgue almost every choice of parameters a_1, a_2, b_1, and b_2 and thus are mild conditions. A combination of assumptions (A) and (B) will enable us to bound the largest Lyapunov exponent away from negative infinity.

LEMMA 3.1. *Under assumption* (B), *the first and second Lyapunov exponents* λ_1 *and* λ_2 *associated with the matrices* $\{T_i\}_{i\geq 1}$ *are given by*

$$\begin{aligned}\lambda_1 &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \ln|m_j + dv_j| + \lim_{n\to\infty} \frac{\ln(m_n^2+v_n^2)}{2n},\\ \lambda_2 &= -\infty\end{aligned} \tag{3.2}$$

if the limit $\lim_{n\to\infty}\ln(m_n^2+v_n^2)/n$ *exists.*

Proof. Recall that $\{z_n' = (x_n, y_n)\}_{n\geq 1}$ is defined recursively by $z_n = A_n z_{n-1} = A_n A_{n-1}\cdot \ldots \cdot A_1 z_0$, where A_i are the affinities defined in (1.2). To compute the Lyapunov exponents, we shall find the singular values of the matrices T_i. Recall that the T_i are two-by-two matrices, whose first row elements are m_i and dm_i and whose second row elements are v_i and dv_i. Let B_n be the product of the first n matrices T_k, that is,

$$B_n = T_n T_{n-1}\cdot \ldots \cdot T_1 = T_n \prod_{j=1}^{n-1}(m_j + dv_j).$$

Let α_{n1} and α_{n2} denote the singular values of the matrix B_n, that is, the nonnegative square roots of the two eigenvalues of $B_n^* B_n$, where B_n^* denotes the transpose of B_n. Then the Lyapunov exponents λ_1 and λ_2 are given by $\lambda_i = \lim_{n\to\infty}\ln\alpha_{ni}/n$ for $i = 1, 2$. If we write $s_{n-1} = \prod_{j=1}^{n-1}(m_j + dv_j)$, then

$$B_n^* B_n = s_{n-1}^2 T_n^* T_n = s_{n-1}^2(m_n^2+v_n^2)\begin{pmatrix} 1 & d\\ d & d^2\end{pmatrix}.$$

Hence, the two eigenvalues of $B_n^* B_n$ are given by $(1+d^2)s_{n-1}^2(m_n^2+v_n^2)$ and 0. Thus, their (nonnegative) square roots are

$$\begin{aligned}\alpha_{n1} &= \sqrt{1+d^2}\,|s_{n-1}|\sqrt{m_n^2+v_n^2},\\ \alpha_{n2} &= 0.\end{aligned}$$

Therefore, we get $\lambda_2 = -\infty$ and

$$\begin{aligned}\lambda_1 &= \lim_{n\to\infty}\frac{1}{n}\ln\alpha_{n1}\\ &= \lim_{n\to\infty}\frac{1}{n}\ln\left[\sqrt{1+d^2}\,|s_{n-1}|\sqrt{m_n^2+v_n^2}\right]\\ &= \lim_{n\to\infty}\frac{\ln(1+d^2)}{2n} + \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}\ln|m_j+dv_j| + \lim_{n\to\infty}\frac{\ln(m_n^2+v_n^2)}{2n}\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\ln|m_j+dv_j| + \lim_{n\to\infty}\frac{\ln(m_n^2+v_n^2)}{2n},\end{aligned}$$

as shown in (3.2). ∎

In the periodic case, there is some integer $P > 0$ so that $T_{i+P} = T_i$ for all integers i, equivalently, $m_{i+P} = m_i$ and $v_{i+P} = v_i$, thus, clearly $\lim_{n\to\infty} \ln(m_n^2 + v_n^2)/n = 0$ and

$$\lambda_1 = \frac{1}{P} \sum_{i=1}^{P} \ln|m_i + dv_i|.$$

Thus, in the subsequent discussion we shall restrict our attention to the *aperiodic* case, where a sequence of lemmas will provide an estimate of a lower bound for the limit in the previous lemma. Once expression (3.2) is bounded from below, λ_1 can be approximated by the following expression in a closed form:

$$\frac{1}{p_\varepsilon} \int_0^{p_\varepsilon} \ln|f(x)|dx$$

for some approximate period p_ε. Crucial ingredients in the proof are the facts that (a) for $1 \le i \le n$, not too many values $|m_i + dv_i|$ are close to zero, and (b) the minimum of these values does not approach zero too fast. A consequence of the remarks following Definition 1.1 is that $\{f(k)\}_{k\ge1}$ is almost periodic. The next result establishes a lower bound for the distance of the values of $f(k)$ from zero.

LEMMA 3.2. *Suppose that assumptions* (A) *and* (B) *hold. Let* $\{T(n)\}_{n\ge1}$ *be an increasing sequence of positive integers such that* $T(n) \to \infty$ *and* $T(n)/n \to 0$ *as* $n \to \infty$. *If there is a subsequence* $\{n_k\}_{k\ge1}$ *such that*

$$|f(n_k)| < [T(n_k)n_k]^{-1} \tag{3.3}$$

for every integer $k \ge 1$, *then* $n_{k+1}/n_k \to \infty$ *as* $k \to \infty$.

Proof. We will prove claim (3.3) by contradiction. Suppose that there is a subsequence $\{n_k\}_{k\ge1}$ such that $|f(n_k)| < [T(n_k)n_k]^{-1}$ for every integer $k \ge 1$ and suppose that there is some integer $w > 0$ such that for arbitrarily large k, we have $n_k < n_{k+1} < wn_k$. Fix such an integer $w > 0$. By the triangle inequality,

$$|f(n_{k+1}) - f(n_k)| < 2\,[T(n_k)n_k]^{-1}.$$

The transversality condition (assumption (A)), along with the assumptions imposed upon the functions s and c, guarantees that f has at most finitely many zeros on any interval of finite length. Thus, without loss of generality we may assume that the small values $f(n_k)$ and $f(n_{k+1})$ correspond to the neighborhood of the same root of f on an interval of length $p = p_\varepsilon$. Note that for sufficiently large k, $|f'(n_k)| > \rho$. If we use the linear approximation $|f(n_{k+1}) - f(n_k)| \approx |f'(n_k)|(n_{k+1} - n_k)(\mathrm{mod}\ p)$ for f, we would obtain, for all sufficiently large k,

$$(n_{k+1} - n_k)(\mathrm{mod}\ p) < \frac{2}{\rho}\,\frac{1}{T(n_k)n_k}.$$

By the uniform continuity of f, the constant

$$C_f = \max_{0\le x\le p} |f'(x)|$$

is positive and finite. Relying on the linear approximation for f in the other direction as well would provide, for every integer $s > n_{k+1}$,

$$|f(s) - f(s - (n_{k+1} - n_k))| \le 2C_f/(\rho T(n_k)n_k) = K/(T(n_k)n_k), \tag{3.4}$$

where we wrote $K = 2C_f/\rho$. Next let I denote the range of the function f on the reals and $|I| < \infty$ its length. Let $J \subset I$ be the vacant space in I, i.e., such that $J \cap \{m_j + v_j :\ \text{for all } j \geq n_k\} = \emptyset$. Since future points in the orbit $\{f(n_i)\}_{i \geq k}$ will be at a distance $\leq K/(T(n_k)n_k)$ from any point of the orbit up to time n_k, by (3.4),

$$\begin{aligned} |J| &\geq |I| - \#\{i :\ 1 \leq i \leq n_{k+1}\} K/(n_k T(n_k)) \\ &= |I| - n_{k+1} K/(n_k T(n_k)) \\ &\geq |I| - w n_k K/(n_k T(n_k)) \\ &= |I| - wK/T(n_k), \end{aligned}$$

where # denotes cardinality. Since $1/T(n_k) \to 0$ as $k \to \infty$, it would then follow that $|J| \geq |I|$ as $k \to \infty$. Thus, $|J| = |I|$. Now observe that f is Lipschitz and the Lebesgue measure is an ergodic invariant measure under the map $\{k/a_2\}_{k \geq 1}$ and $\{k/b_2\}_{k \geq 1}$ [10]. Consequently, each interval of positive one-dimensional Lebesgue measure must be hit infinitely often by values of f. This is a contradiction. Since w was arbitrary, it follows that $n_{k+1}/n_k \to \infty$ as $k \to \infty$. $\square$

The number of values of $f(k)$ close to zero is bounded as follows.

LEMMA 3.3. *Impose assumptions* (A) *and* (B) *and let* $\{T(n)\}_{n \geq 1}$ *be a sequence as described in Lemma* 3.2. *Then there is some positive finite constant* C *such that for all sufficiently large* n,

$$\#\{1 \leq k \leq n :\ |f(k)| < 1/T(n)\} \leq Cn(1 + \varepsilon_n)/T(n), \tag{3.5}$$

where # denotes cardinality and $\varepsilon_n \to 0$ *exponentially fast with increasing* n.

Proof. Observe that $x \mapsto x + 1/a_2 \pmod{p_1}$ and $x \mapsto x + 1/b_2 \pmod{p_2}$ are ergodic (see, e.g., [10, p. 49]) and that the one-dimensional Lebesgue measure is invariant under these two transformations. Let $A_n = \{1 \leq k \leq n :\ |f(k)| < 1/T(n)\}$. Since f is Lipschitz, there is a positive finite constant C such that for all sufficiently large n, the expected cardinality of A_n, under the normalized Lebesgue measure, is bounded above by $Cn/T(n)$. Fix some $\varepsilon > 0$. A routine argument resting on the central limit theorem, applied to the sum of i.i.d. Bernoulli random variables, yields that there is some constant $c_2 > 0$ such that, for all sufficiently large n,

$$\mathbf{P}(\#A_n > Cn(1 + \varepsilon)/T(n)) \leq e^{-n\varepsilon^2 c_2},$$

where $\mathbf{P}$ denotes the uniform distribution (or normalized Lebesgue measure). Since $\varepsilon > 0$ was arbitrary, this completes the proof. $\square$

Next, if for every $\eta > 0$ we define the set

$$B = B_\eta = \{1 \leq k \leq n :\ |f(k)| < \eta\} \tag{3.6}$$

of "bad" integers between 1 and n, then clearly,

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n-1} \ln|m_j + dv_j| &= \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n-1} \ln|m_j + dv_j| \left\{1_{\{k \in B\}} + 1_{\{k \notin B\}}\right\} \\ &\geq \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n-1} \ln|m_j + dv_j| \, 1_{\{k \in B\}} + \ln \eta, \end{aligned} \tag{3.7}$$

where $1_{\{\cdot\}}$ denotes the indicator function. It remains to bound below the sum on the right-hand side of inequality (3.7).

LEMMA 3.4. *Suppose that assumptions* (A) *and* (B) *hold. Then for sufficiently small* $\eta > 0$, *we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \ln|m_j + dv_j|\, 1_{\{k\in B_\eta\}} \geq \omega_f, \tag{3.8}$$

where $r_L = \#\{0 \leq x \leq L : \ f(x) = 0\} < \infty$ *for some integer* $L > 0$ *and* $\omega_f = \min(\eta, \eta/\rho) r_L(\ln \eta - 1)/L$.

Proof. Again let $\{T(n)\}_{n\geq 1}$ be a sequence as described in Lemma 3.2. Since the Lebesgue measure is an ergodic invariant measure for the transformation $x \mapsto x + 1 \,(\mathrm{mod}\ p_i)$, $i = 1, 2$, we have, for every $\eta > 0$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} \ln|f(k)|\, 1_{\{k\in B_\eta\}} = \lim_{n\to\infty} \frac{1}{n} &\left\{ \int_0^{n-1} \ln|f(x)|\, 1_{\{1/T(n)\leq |f(x)|<\eta\}}\, dx \right.\\ &\left. + \sum_{k=1}^{n-1} \ln|f(k)|\, 1_{\{|f(k)|<1/T(n)\}} \right\}.\end{aligned} \tag{3.9}$$

We first prove that the limit of the second term in (3.9) is zero. Pick $\delta > 0$ and some $L = L(\delta)$ so that each interval of reals of length L contains at least one translation number τ_δ for f. By Lemma 3.2, the gap between two integers n_k and n_{k+1} such that $f(n_k) \leq [T(n_k)n_k]^{-1}$ and $f(n_{k+1}) \leq [T(n_{k+1})n_{k+1}]^{-1}$ equals $n_k(w_k - 1)$, where $w_k \to \infty$ as $k \to \infty$. Now pick k large enough such that there is an interval of integers of length L, say, $n_0, n_0 + 1, n_0 + 2, \ldots, n_0 + L$ such that for every integer $n_k < n_0 \leq j \leq n_0 + L < n_{k+1}$, we have

$$|f(j)| > [T(j)j]^{-1}.$$

Combining this with Lemma 3.3 and using the almost periodicity of f, we find that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n-1} \ln|f(k)|\, 1_{\{|f(k)|<1/T(n)\}} &= \frac{n-1}{L\,n} \sum_{k=n_0+1}^{n_0+L} \ln|f(k)|\, 1_{\{|f(k)|<1/T(n)\}} + c\delta\\ &\geq -[\ln n + \ln T(n)]C(1 + \varepsilon_n)/T(n) + c\delta\end{aligned}$$

for some constants $0 < C, c < \infty$, where $\varepsilon_n \to 0$ exponentially fast. Now choose $T(n)$ such that $\ln n/T(n) \to 0$ as $n \to \infty$. This shows that, for sufficiently large n, the left-hand side of the last displayed equation is larger than or equal to zero because δ was arbitrary. Since clearly it is also nonpositive, it must be exactly zero in the limit.

Second, we will bound the first term of the right-hand side in (3.9). Recall that $f'(x) \leq C_f$ and that $f(x) = 0$ implies $|f'(x)| > \rho$. Thus, $f(x) < \delta$ for some sufficiently small δ implies $|f'(x)| \geq c\rho$ for some positive finite constant c, with $c \to 1$ as $\delta \to 0$. Now suppose that x_0 is a zero of f. Then for $|x - x_0| < \delta$, we obtain $f(x) = f'(x_0)(x - x_0) + o(\delta)$. Below we carry out only the steps for $x - x_0 > 0$ and $f'(x_0) > 0$. Minor modifications establish the case $x - x_0 < 0$. Write $Z_L = Z_f \cap [0, L]$. Straightforward calculus yields

$$
\begin{aligned}
\int_0^{n-1} \ln|f(x)|\, 1_{\{|f(x)|\in[1/T(n),\eta)\}}\, dx &= \frac{n-1}{L} \int_0^{L} \ln|f(x)|\, 1_{\{|f(x)|\in[1/T(n),\eta)\}}\, dx + o(\delta) \\
&= \frac{n-1}{L} \sum_{x_0\in Z_L} \int_0^{L} \ln|f(x)| \\
&\quad \cdot 1_{\{|f'(x_0)(x-x_0)|\in[1/T(n),\eta)\}}\, dx + o(\delta) \\
&= \frac{n-1}{L} \sum_{x_0\in Z_L} \int_0^{L} \ln[|f'(x_0)(x-x_0)|] \\
&\quad \cdot 1_{\{|f'(x_0)(x-x_0)|\in[1/T(n),\eta)\}}\, dx + o(\delta) \\
&= \frac{n-1}{Lf'(x_0)} \sum_{x_0\in Z_L} \int_{-f'(x_0)x_0}^{f'(x_0)(L-x_0)} \ln y \\
&\quad \cdot 1_{\{y\in[1/T(n),\eta)\}}\, dy + o(\delta) \\
&= \frac{n-1}{Lf'(x_0)} \sum_{x_0\in Z_L} \int_{1/T(n)}^{f'(x_0)(L-x_0)\wedge\eta} \ln y\, dy + o(\delta).
\end{aligned}
$$

We may assume $L \neq x_0$ (because otherwise the interval length L may be chosen differently). Pick $\eta \leq \rho \min_{x_0\in Z_L}(L-x_0)$, divide both sides by n, and take the limit

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \int_0^{n-1} \ln|f(x)|\, 1_{\{|f(x)|\in[1/T(n),\eta)\}}\, dx &\geq \lim_{n\to\infty} \frac{1}{n}\, \frac{n-1}{L} \sum_{x_0\in Z_L} \{\min(\eta,\eta/\rho)(\ln\eta - 1) \\
&\qquad + (\ln T(n)+1)/(T(n)\rho)\} \\
&= \frac{r_L}{L} \min(\eta,\eta/\rho)(\ln\eta-1),
\end{aligned}
$$

as desired. ∎

This completes the proof of Theorem 1.3.

Corollary 3.5. *Under assumptions* (A) *and* (B),

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} \ln|m_j + dv_j| \geq K_f > -\infty, \tag{3.10}
$$

where $K_f = \ln\eta + r_L \min(\eta,\eta/\rho)(\ln\eta-1)/L$, and r_L, η, L, and ρ are positive finite constants depending on the particular function f. Moreover, in the aperiodic case, if the limit

$$
\lim_{n\to\infty} \frac{\ln(m_n^2+v_n^2)}{n}
$$

exists, then it equals 0, *and the largest Lyapunov exponent is bounded below, i.e., $\lambda_1 \geq K_f$. Fix $\varepsilon > 0$. Then*

$$
\left|\lambda_1 - \frac{1}{p_\varepsilon} \int_0^{p_\varepsilon} \ln|f(x)|\, dx\right| \leq \varepsilon, \tag{3.11}
$$

where p_ε is the approximate period defined in Definition 1.2. *The limit $\lim_{n\to\infty} \ln(m_n^2 + v_n^2)/n$ exists for Lebesgue almost every choice of values for a_2 and b_2.*

Proof. Combining (3.7) and Lemma 3.4 yields (3.10). Finally, (3.11) is an immediate consequence of Lemma 3.1. ∎

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