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SYMBOLIC DYNAMICS FOR IFS ATTRACTORS

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Abstract

It has recently been shown that a modified iterated function system (IFS) is capable of generating closed orbits which undergo bifurcation and transition to a chaotic regime as control parameters are varied.[1,2] Here we show that driving such an IFS by a partition of itself creates maps which can be characterized by a symbolic dynamics. Forbidden words are determined for this dynamics under various parameter values, and the implications of this mapping are discussed.

1. INTRODUCTION

It has been known for some time that iterated function systems (IFS) generate fractal images[3–5] and can also be used to generate curves, surfaces and wavelets.[6,7] In a recent work,[1,2] we have shown that a modified IFS can generate images resembling the closed orbits, bifurcations and attractors found in classical chaotic systems such as the Lorenz attractor or the Rossler band.

A standard IFS consists of N affine transformations of the form:

$$x(n+1) = a_i x(n) + b_i y(n) + c_i \tag{1a}$$

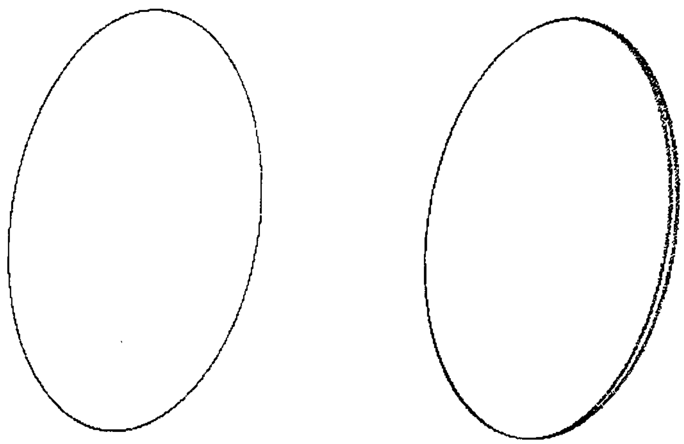
$$y(n+1) = d_i x(n) + e_i y(n) + f_i \tag{1b}$$

where $i = 1, \ldots, N$. Each successive point is generated from the previous point by means of one of the N transformations, selected randomly, with probability p_i. Suppose, however, that we were to modify such an IFS by introducing discontinuous forcing terms dependent on the number of the iteration. For example, let us set $N = 2$ and $p_1 = p_2 = 0.5$, and let:

$$x(n+1) = s_x \sin(n/w_x)[a_i x(n) + b_i y(n)] + c_i \tag{2a}$$

$$y(n+1) = s_y \cos(n/w_y)[d_i x(n) + e_i y(n)] + f_i \tag{2b}$$

where s_x, s_y, w_x and w_y are constants which do not change from transformation A to transformation B, and the other parameters, a_i, b_i, c_i, d_i, e_i, f_i,



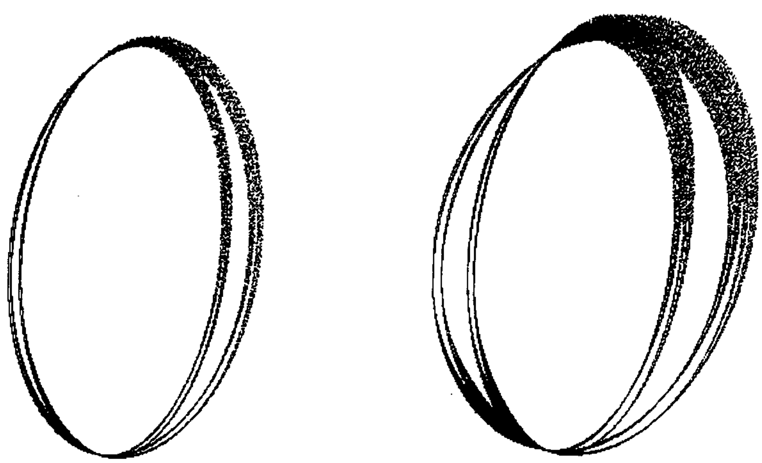
1.0 1.0 1.0 1.0
1.0 1.0 1.0 1.0
(degenerate case)

(a)

0.9 1.0 1.0 1.0
1.0 1.0 1.0 1.0

(b)

Fig. 1(a) and 1(b) Degeneracy breaking simulates bifurcation. A "period one orbit" is shown in 1(a), generated by the iterated function system in Eqs. 2(a) and 2(b). Parameters are set as $c_i = 0.1$, $f_i = 0.2$, $s_x = 0.3$, $s_y = 0.45$, $w_x = w_y = 0.16$. These parameters will be held constant throughout the series of orbits and attractors shown in Fig. 1. The coefficients of the x and y terms in transformations A and B (a_i, b_i, d_i, e_i in Eq. 2) are shown in matrix form to be adjacent to each "orbit". As with the images shown in the rest of Fig. 1, this picture was produced by setting $x_{\text{init}} = y_{\text{init}} = 100$, discarding transients, and then plotting the image in the xy domain. The x and y values were multiplied by a scale factor of 833 before plotting.



0.7 0.9 1.0 1.0
1.0 1.0 1.0 1.0

(c)

0.7 0.9 1.0 1.3
1.0 1.0 1.0 1.0

(d)

Fig. 1(c) through 1(l) As the coefficients of the x and y terms (shown in matrix form) continue to vary, the IFS generates images which resemble multiply periodic orbits and chaotic attractors.

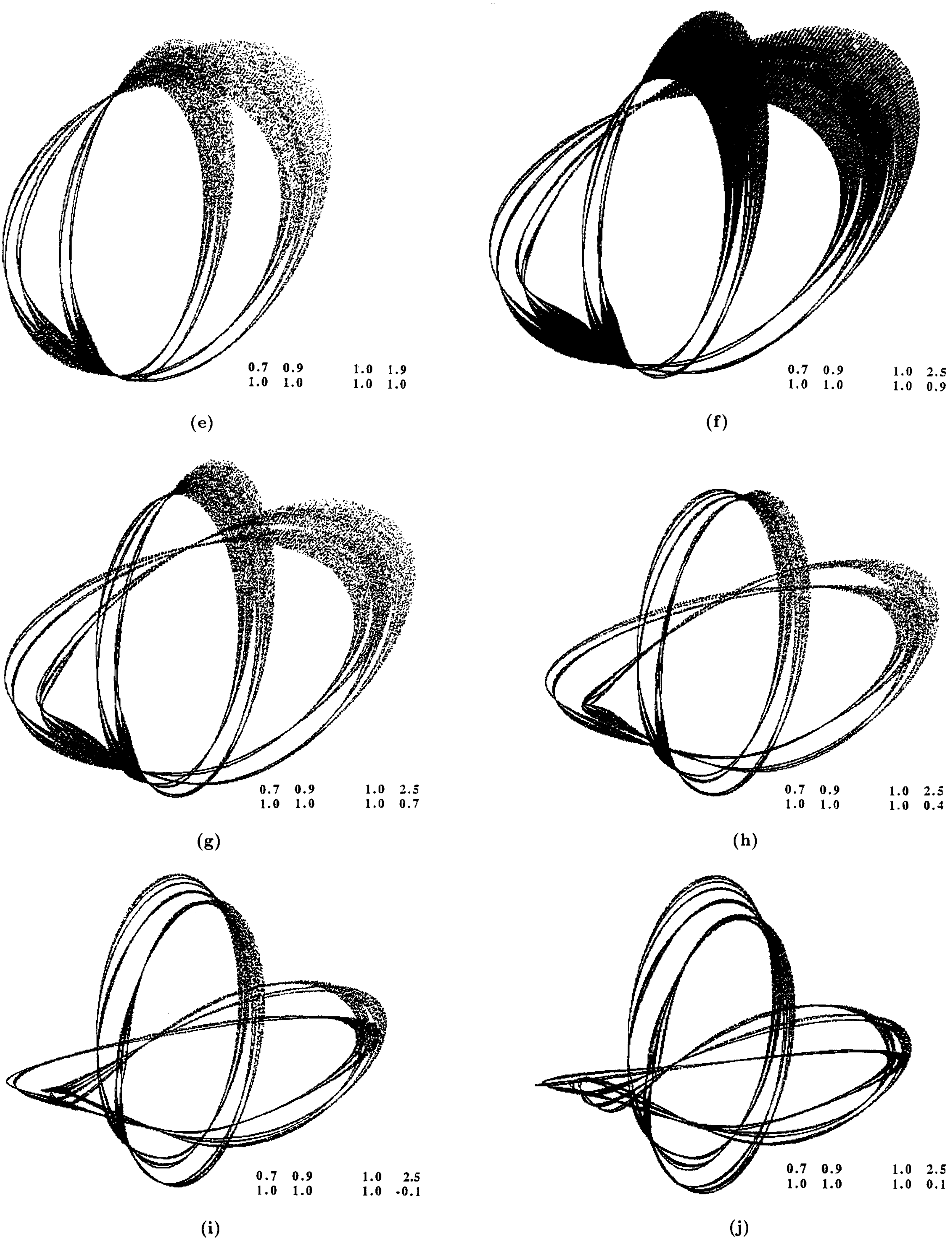
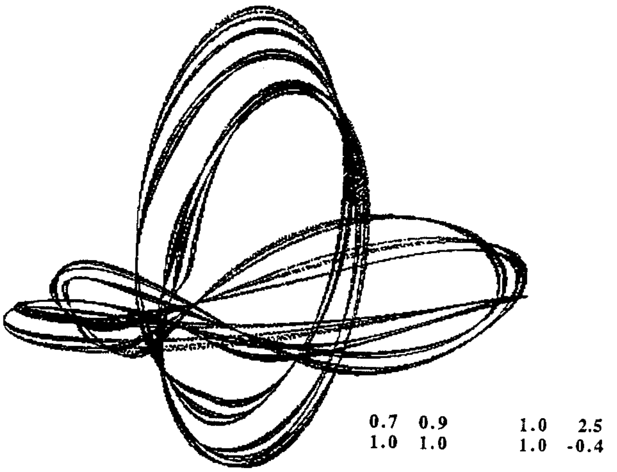
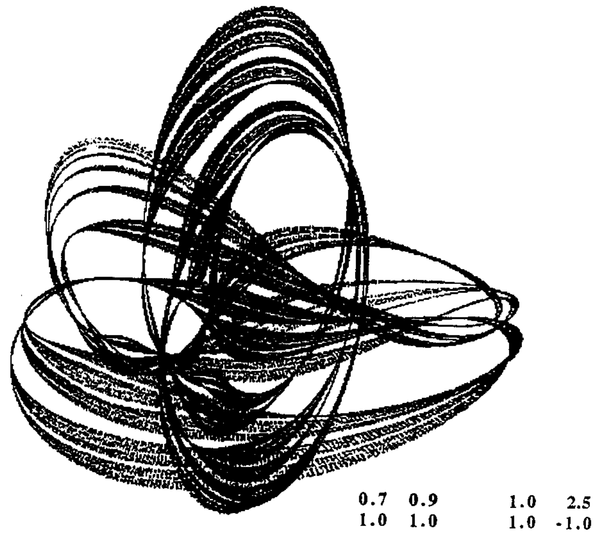


Fig. 1(c) through 1(l) (*Continued*)



(k)



(l)

Fig. 1(c) through 1(l) *(Continued)*

$i = 1, \ldots, N$, are unique to each transformation. We will refer to the transformation for which $i = 1$ as transformation A and that for which $i = 2$ as transformation B (this will facilitate later notation, as will become evident below).

Consider a case in which $s_x = 0.3$, $s_y = 0.45$, $w_x = w_y = 0.16$ $c_i = 0.1$, $f_i = 0.2$, and $a_i = b_i = d_i = e_i = 1.0$, $i = 1, \ldots, N$. In this case transformation A is identically equal to transformation B. When we select initial points $x_{\text{init}} = y_{\text{init}} = 100$ and plot successive iterates of the algorithm in the xy plane (after discarding transients), we obtain a closed curve such as that shown in Fig. 1(a). If we now change $a_1 = a_2 = 1.0$ to $a_1 = 0.9$, $a_2 = 1.0$, we obtain the image shown in Fig. 1(b), essentially resembling a "period two orbit " generated by a "bifurcation" from the "period one orbit" in Fig. 1(a). The image shown in Fig. 1(a), with all parameters in transformation A equal to those in B, represents in effect a degenerate case. By breaking the degeneracy (i.e., by making A no longer coinciding identically with B), we simulate "bifurcation" to a multiply periodic orbit. If the parameters are further varied, we obtain further bifurcations and deformation of the orbits as we approach a "chaotic" regime, as shown in the rest of Fig. 1. The only parameters being varied are a_i, b_i, d_i, e_i (the coefficients of the x and y terms in each transformation). These are shown in matrix form adjacent to each "attractor".

Of course, it is important to emphasize that the terms "period one" and "period two orbit" are being used loosely, and in no way represent period k solutions of the system in Eq. (2). These images are collections of points which *look* like orbits, and we describe them as such from an aesthetic, rather than a rigorous mathematical, standpoint.

It is obviously of great interest to characterize the "chaotic" behavior generated by this IFS. Is it in fact a dynamical system? If so, how do the dynamics of IFS attractors relate to the dynamics of attractors generated by nonlinear differential equations? We have recently shown that *embedded orbits* may be isolated within such attractors, and that the self-linking numbers of these "embedded orbits" may be used to generate a *symbolic classification* of the system as it progresses from an apparently "periodic" to a "chaotic" regime.[17]

The possibility of generating a symbolic classification for IFS attractors raises the question of whether we can obtain an actual *symbolic dynamics* for IFS attractors, analogous to the symbolic dynamics used to classify chaotic behavior in nonlinear systems,[8–10] and piecewise linear maps.[11] This would take us a long way toward unravelling the structure of IFS attractors and gaining some understanding of their relationship to classical nonlinear dynamical systems. In what follows, we present a scheme for constructing such a symbolic dynamics.

2. AN IFS DRIVEN BY A PARTITION OF ITSELF

Symbolic dynamics provides a technique for revealing, albeit in a "coarse-grained" way, the underlying dynamics of a complex system. A typical approach to constructing a symbolic dynamics is to divide a chaotic system at a critical point and assign symbol sequences to points on the map according as they fall on one side or the other of the critical point. The set of all allowable strings of symbols provides a "language" which can be a powerful tool for revealing the underlying dynamics of the system it describes.

For IFS attractors, an obvious approach to the development of a symbolic dynamics might be to

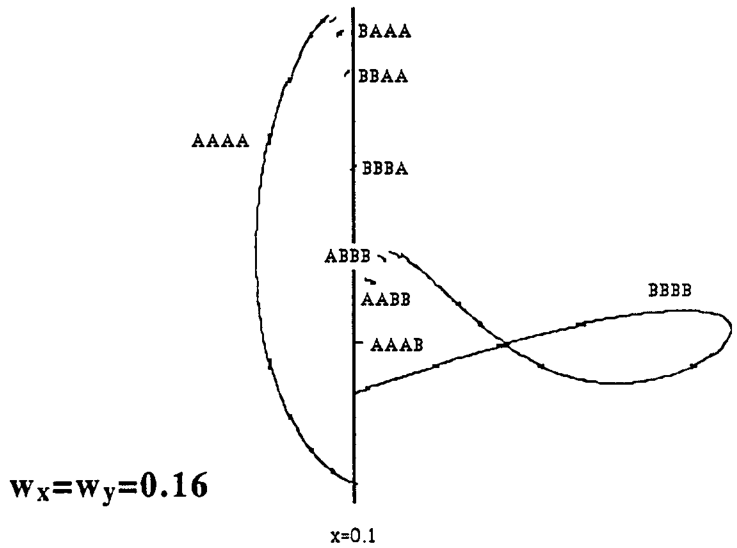


Fig. 2 Symbolic dynamics for an IFS attractor. Using parameters corresponding to the attractor in Fig. 1(l), we have driven the IFS, rather than by a random sequence, by a partition of itself around the axis $x = 0.1$. The fragmented sections of this curve can be labeled by symbol strings as described in the text. Forbidden words corresponding to this dynamics are shown in Table 1.

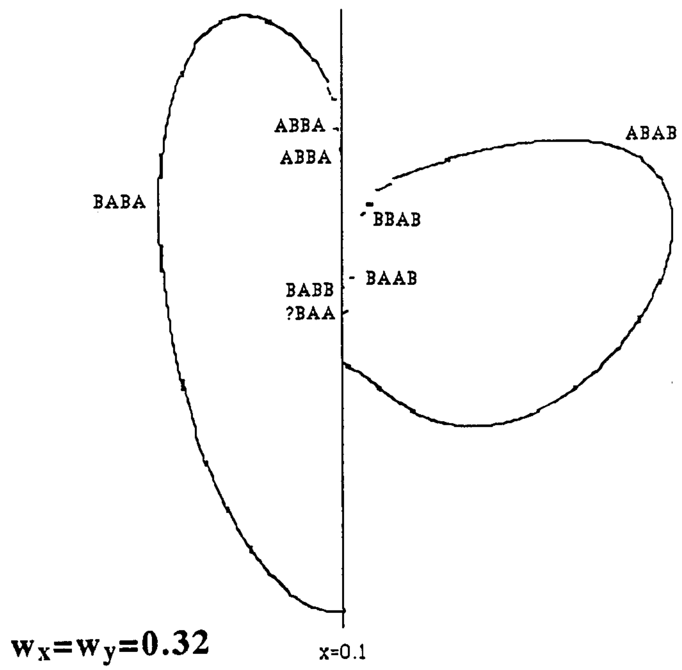


Fig. 3 Symbolic dynamics for an IFS attractor with $w_x = w_y = 0.32$. All other parameters are identical to those in Fig. 1(l) (and hence Fig. 2). Forbidden words are shown in Table 2.

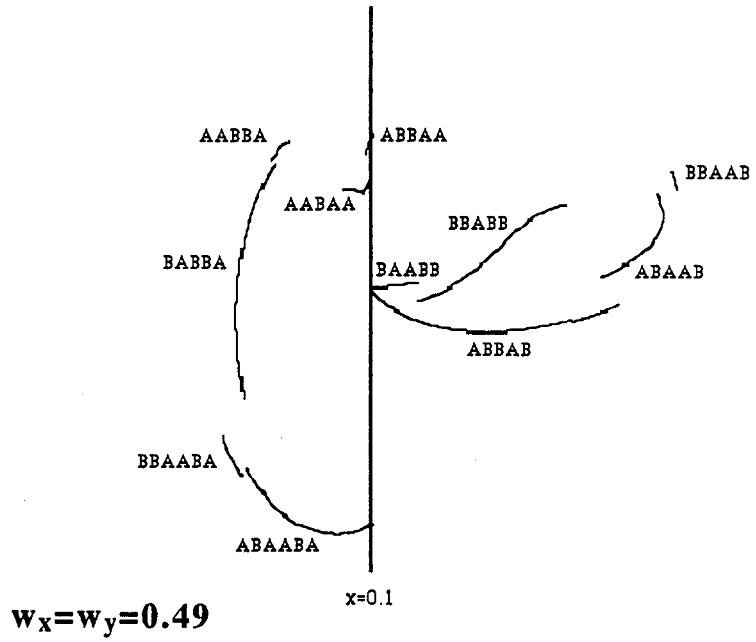


Fig. 4 Symbolic dynamics for an attractor identical to that in Fig. 1(l) except that $w_x = w_y = 0.49$. Forbidden words are listed in Table 3.

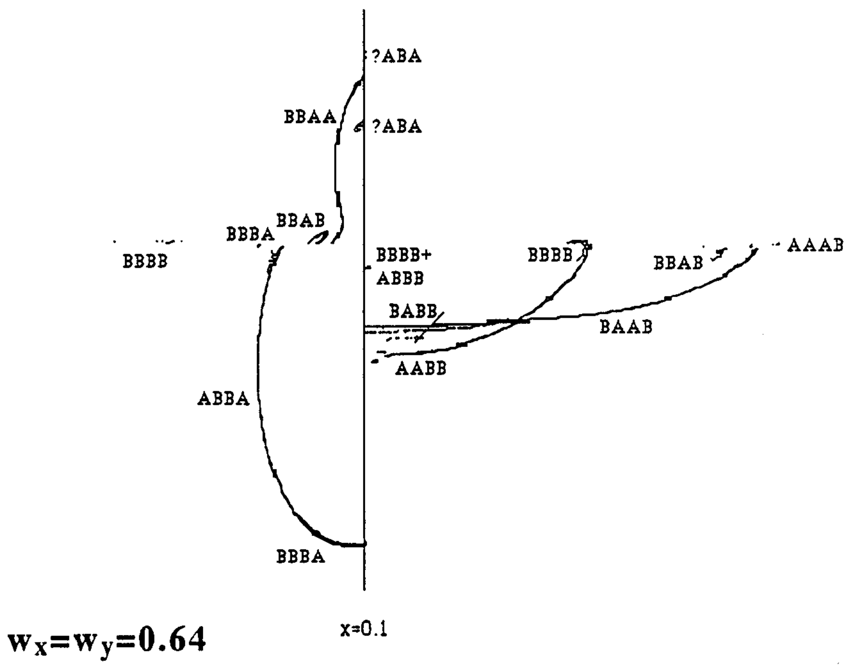


Fig. 5 Symbolic dynamics when $w_x = w_y = 0.64$ and all other parameters are set as in Fig. 1(l). Forbidden words are shown in Table 4.

label points *according to the sequence of transformations generating them.* However, all possible combinations of A and B generate points in the xy domain of IFS attractors.[17] Thus there are no excluded sequences of symbol strings, or no "forbidden words". Without forbidden words, we have no set of rules defining a "language" on the alphabet $\{A, B\}$, and thus no useful symbolic dynamical description of our system. Rule-restricted languages on $\{A, B\}$ may be obtained, however, by studying the dynamics of an IFS attractor *filtered through a partition of itself*, as we shall see below.

The IFS attractors we have shown in Fig. 1 are, of course, generated by a random sequence of the

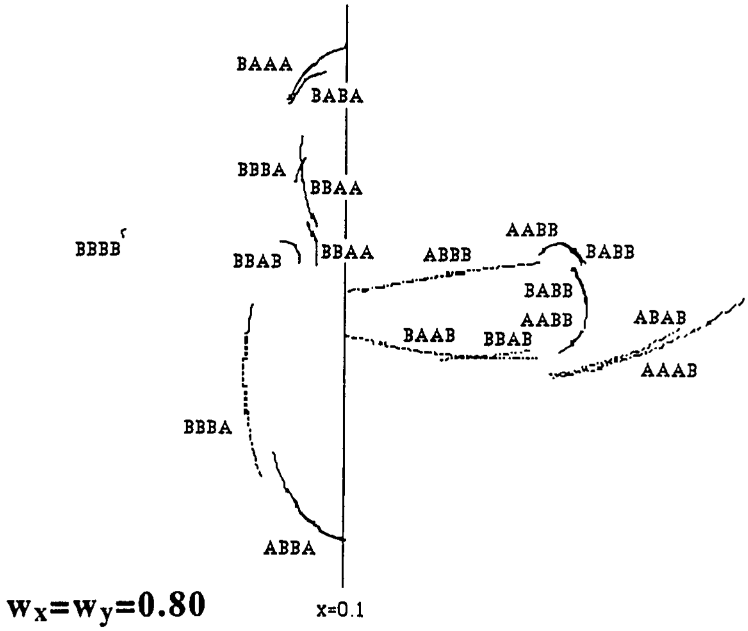


Fig. 6 Symbolic dynamics when $w_x = w_y = 0.80$, with forbidden words listed in Table 5. A comparison of Figs. 2–6 and Tables 1–5 shows that the dynamics is strongly dependent on the value of w_x and w_y even when all other parameters are equal.

transformations A and B. This is the standard type of IFS algorithm, used to generate most fractal images such as the Sierpinski gasket, etc. Recently, however, nonlinear dynamicists have explored the possibility of driving IFS algorithms by nonrandom sequences.[5,12–14] Explorations of nonrandom driving sequences (driven IFS's, in Barnsley's terminology; language-restricted IFS's in Prusinkiewicz's) have revealed some interesting dynamical behavior (See Ref. 13 for example, on the Sierpinski gasket being driven by a number sequence obtained from the chaotic region of the logistic map, with quite interesting results!), and raised interesting questions regarding the role of randomness in generating fractal images and the degree of statistical randomness in chaotic sequences. This method of driving an IFS by a nonrandom sequence can be used to obtain a symbolic dynamics for IFS attractors, as we shall see. When our IFS algorithm is run in the standard random fashion, using parameters in the range of the images shown in Fig. 1, points are generated whose x values fall both above and below the axis $x = 0.1$. Approximately 50% of points fall on either side of this x-coordinate, so that we may consider it as a statistical partition of the system. In addition, some singular points for a system where $w_x = w_y = Q/2\pi$ lie on this line, Q some integer.[15] All the $w_x = w_y$ values we have used here are just slightly larger than $Q/2\pi$. So that $x = 0.1$ appears to be at or near a natural as well as a statistical partition of the space.

Now, as we run the randomized algorithm, we obtain a sequence of numbers which lie above or below the line x=0.1. These numbers may be characterized by a symbolic sequence according as they fall above or below this partition. Suppose now we feed this sequence into a new run of the algorithm *as a driving sequence* rather than using a random number generator.[16] We will use the partition to generate a driving sequence as follows:

if $x < 0.1$ ---> transformation A ,

if $x > 0.1$ or $x = 0.1$ ---> transformation B .

Running the algorithm now, with this partition-generated driving sequence, we obtain xy domain plots such as those shown in Figs. 2–6 rather than "attractors" such as those shown in Fig. 1. Note that these images are often discontinuous across the partition (shown as the y-axis at $x = 0.1$) and break into a number of disconnected segments. These segments may be described by symbol sequences as in a symbolic dynamics, as we shall show.

3. SYMBOLIC DYNAMICS FOR IFS ATTRACTORS

If one looks at how the points in Figs. 2–6 fall one either side of the partition line as a function of the sequence of transformations which produces the point, one notices that whenever transformation A has just been applied, the ensuing point usually lands on one of the branches on the left side of the partition (i.e., $x < 0.1$). When transformation B has just been applied, the ensuing point usually lands on one of the branches on the right side of the partition ($x > 0.1$ or $x = 0.1$). Thus the points on the left of $x = 0.1$ may be characterized by the string A, those on the right by the string B.

Note that there are a few segments which do not absolutely follow this rule, such as the segment denoted as ABBAA at the top of Fig. 4 ($w_x = w_y = 0.49$) or that denoted as BBBA in Fig. 2 ($w_x = w_y = 0.16$). In such cases, where a point generated from transformation A lands on the right side of the partition, we denote it as string A, i.e., the generating transformation takes precedence over the informal rule of left side vs. right side. This rule, in any event, holds only for the most recent symbol in the string, as we shall see below. We speculate that variation from this rule may be due to the inaccuracy of $x = 0.1$ as a perfect partition (i.e., some $x + \partial x$ may be a better choice).

Now, suppose we look at all points appearing when transformation A has been employed, *preceded by another application of transformation A*. In Fig. 2 ($w_x = w_y = 0.16$), we see that such points may land on the large elliptical arc or any of its satellite points clustering at the top of the picture. If we look for points appearing after an application of transformation A, *preceded by an application of transformation B*, we see that such points land only on the small line fragment which appears to cross the partition. This fragment may be denoted by the symbol string BA, and the points on the elliptical arc and its cluster of satellites may be denoted as string AA. Note that we choose a convention whereby the most recently employed transformation is located at the far right, and earlier transformations extend leftward.

The segments may be broken down further, and each segment can be labeled uniquely with a symbol string. This assignment of strings could be continued ad infinitum, with smaller segments being given longer and longer string assignments, but for clarity of presentation we have extended our dynamics only to strings of length 4 [or, for Fig. 4 ($w_x = w_y = 0.49$) strings of length 5]. In general, strings of these lengths seem sufficient to uniquely label each visibly separate line segment within the image.

As one can see from examining Figs. 2–6, there are various sequences of transformations which do not fall on any of the curved fragments in a given figure, and thus for each figure some symbol strings do not occur in our language on $\{A, B\}$. These are forbidden words. A list of the forbidden words for a particular parameter value allows us to define a formal language on the alphabet $\{A, B\}$, characterizing the dynamics of our IFS system for that parameter value. Forbidden words corresponding to the pictures in Figs. 2–6 are listed in Tables 1-5, respectively, up to a length of 4 (or length 5, for $w_x = w_y = 0.49$). We list all forbidden words (including those formed by leftward extensions of shorter

Table 1

$w_x = w_y = 0.16$	
String Length	**Forbidden Words**
1	----
2	----
3	ABA
3	BAB
4	AABA
4	BABA
4	ABAB
4	BBAB
4	ABBA
4	ABAA
4	BAAB
4	BABB

Table 2

$w_x = w_y = 0.32$	
String Length	**Forbidden Words**
1	----
2	----
3	AAA
3	BBB
4	AAAA
4	BBBB
4	BAAA
4	ABBB
4	BBBA
4	AAAB
4	BBAA
4	AABB

Table 3

$w_x = w_y = 0.49$	
String Length	**Forbidden Words**
1	----
2	----
3	AAA
3	BBB
4	AAAA
4	BBBB
4	BABA
4	BBBA
4	AAAB
4	ABAB
4	ABBB
4	BAAA
5	[all 5-letter words formed by leftward extensions of the forbidden 4-letter words, plus the following 5-letter words]
5	AAABA
5	BBBAB
5	AAABB
5	BBBAA
5	ABABB
5	BABAA

Table 4

$w_x = w_y = 0.64$	
String Length	**Forbidden Words**
1	----
2	----
3	AAA
4	AAAA
4	ABAA
4	ABAB

Table 5

$w_x = w_y = 0.80$	
String Length	**Forbidden Words**
1	----
2	----
3	AAAA
4	ABAA
4	AABA
4	BBBB

forbidden words) up to length 4, and principal forbidden words only of length 5. Obviously, new principal forbidden words may appear in strings of length greater than 4 or 5. Thus the set of rules describing the formal language for each parameter value is not necessarily complete.

The existence of forbidden words gives the language itself a fractal structure.[12] Since the number of forbidden words of each length varies from parameter value to parameter value, we may conclude that the fractal dimension of the language varies with w_x and w_y. It still remains to calculate these fractal dimensions and determine whether there is any pattern in the way the dimension depends on the parameters w_x and w_y. Further, it is possible that a different choice of partition might affect the fractal dimension of the language on $\{A, B\}$.

4. CONCLUSIONS: THE INTERPRETATION OF FORBIDDEN WORDS

The ability to generate a symbolic dynamics for IFS attractors is a powerful tool for the analysis of these attractors as dynamical systems, and may provide a useful point for comparison of IFS attractors with chaotic attractors generated by more standard means (i.e., systems of nonlinear differential equations). An interesting starting point for such a comparison might be the meaning of forbidden words in IFS attractors as against in more typical chaotic systems. Whereas in nonlinear dynamical systems forbidden words represent forbidden transitions from one region of the system to another, in IFS symbolic dynamics forbidden words represent sequences of transformations which for some reason fail to generate segments in the xy plane when the IFS is filtered through a partition of itself. The implications of this difference in the interpretation of forbidden words remain to be investigated.

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This paper is dedicated to the memory of the author's mother.

REFERENCES

1. S. Bahar, "Chaotic orbits and bifurcation from a fixed point generated by an iterated function system," *Chaos, Solitons and Fractals* **5**(6), 1001–1006 (1995).
2. S. Bahar, "Further studies of bifurcations and chaotic orbits generated by iterated function systems," *Chaos, Solitons and Fractals* *7*(1), 41–47 (1996).
3. M. F. Barnsley and S. Demko, "Iterated function systems and the global construction of fractals," *Proc. R. Soc. Lond.* **A399**, 243–275 (1985).
4. M. F. Barnsley, V. Ervin, D. Hardin and J. Lancaster,. "Solution of an inverse problem for fractals and other sets," *Proc. Natl. Acad. Sci. USA* **83**, 1975–1977 (1986).
5. M. F. Barnsley, "Fractals Everywhere," 2nd Edition. *Academic Press Inc.*, New York (1993).
6. M. A. Berger, "Random affine iterated function systems: Curve generation and wavelets," *SIAM Review* **34**, 361–385 (1992).
7. P. R. Massopust, "Smooth interpolating curves and surfaces generated by iterated function systems," *J. Analysis and Applications (Z. Anal. Anwendungen)* **12**, 201–210 (1993).
8. V. Daniels, M. Vallieres and J.-M. Yuan, "Chaotic scattering on a double well: Periodic orbits, symbolic dynamics, and scaling," *Chaos* **3**(4), 475–485 (1993).
9. H.-P. Fang and B.-L. Hao, "Symbolic dynamics of the Lorenz equations," *Chaos, Solitons and Fractals* **7**(2), 217–246 (1996).
10. P. Cvitanovic, G. H. Gunaratne and I. Procaccia, "Topological and metric properties of Henon-type strange attractors," *Phys. Rev. A.* **38**(3), 1503–1520 (1988).
11. T. Tel, "Invariant curves, attractors and phase diagram of a piecewise linear map with chaos," *J. Stat. Phys.* **33**, 195–221 (1993).
12. M. Frame, "A fractal representation of grammatical complexity in time series," *Manuscript in Preparation* (1996).
13. I. Stewart, "Order within the chaos game?" *The Dynamics Newsletter, Santa Cruz, CA* **3**, 4–9 (1989).
14. P. Prusinkiewicz and M. S. Hammel "Escape-time visualization method for language-restricted iterated function systems," in *Fractal Geometry and Computer Graphics*, J. L. Encarnacao, H.-O. Peitgen, G. Sakas, G. Englert, eds. Springer Verlag, Berlin, 24-43 (1992).
15. S. Bahar, *Unpublished Data* (1996).
16. M. Frame, *Personal Communication* (1996).
17. S. Bahar, "Orbits embedded in IFS attractors," *International Journal of Bifurcation and Chaos*, to appear (March 1997).