A one-map two-clock approach to teaching relativity in introductory physics

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This paper presents some ideas which might assist teachers incorporating special relativity into an introductory physics curriculum. One can define the proper-time/velocity pair, as well as the coordinate-time/velocity pair, of a traveler using only distances measured with respect to a single "map" frame. When this is done, the relativistic equations for momentum, energy, constant acceleration, and force take on forms strikingly similar to their Newtonian counterparts. Thus high-school and college students not ready for Lorentz transforms may solve relativistic versions of any single-frame Newtonian problems they have mastered. We further show that multi-frame calculations (like the velocity-addition rule) acquire simplicity and/or utility not found using coordinate velocity alone. From physics-9611011 (xxx.lanl, NM, 1996).

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I. INTRODUCTION

Since the 1920's, it has been known that classical Newtonian laws depart from a description of reality, as velocity approaches the speed of light. This divergence between Newtonian and relativistic physics was one of the most remarkable discoveries of this century. Yet, more than 70 years later, most American high schools and colleges teach introductory students Newtonian dynamics, but then fail to teach students how to solve those same problems relativistically. When relativity is discussed at all, it often entails only a rather abstract introduction to Lorentz transforms, relegated to the back of a chapter (or the back of a book).

This is in part due to the facts that: (i) Newton's laws work so well in routine application, and (ii) introductions to special relativity (often patterned after Einstein's introductions to the subject) focus on discovery-philosophy rather than applications. Caution, and inertia associated with old habits, may play a role as well. It is nonetheless unfortunate because teaching Newtonian solutions without relativistic ones at best leaves the student's education out-of-date, and deprives them of experiences which might spark an interest in further physics education. At worst, partial treatments may replace what is missing with misconceptions about the complexity, irrelevance, and/or the limitations of relativity in the study, for example, of simple things like uniform acceleration.

Efforts to link relativistic concepts to classical ones have been with us from the beginning. For example, the observation that relativistic objects behave at high speed as though their inertial mass increases in the $\overrightarrow{p} = m\overrightarrow{v}$ expression, led to the definition (used in many early textbooks[1]) of relativistic mass $m' \equiv m\gamma$. Such efforts are worthwhile because they can: (A) potentially allow the introduction of relativity concepts at an earlier stage in the education process by building upon already-mastered classical relationships, and (B) find what is fundamentally true in both classical *and* relativistic approaches. The concepts of transverse (m') and longitudinal ($m'' \equiv m\gamma^3$) masses have similarly been used[2] to preserve relations of the form $F_x = ma_x$ for forces perpendicular and parallel, respectively, to the velocity direction.

Unfortunately for these relativistic masses, no deeper sense has emerged in which the mass of a traveling object either changes, or has directional dependence. Such masses allow familiar relationships to be used in keeping track of non-classical behaviors (item A above), but do not (item B above) provide frame-invariant insights or make other relationships simpler as well. Hence majority acceptance of their use seems further away now[3] than it did several decades ago[4].

A more subtle trend in the literature has been toward the definition of a quantity called proper velocity[5,6], which can be written as $\overrightarrow{w} \equiv \gamma\overrightarrow{v}$. We use the symbol w here because it is not in common use elsewhere in relativity texts, and because w resembles γv from a distance. This quantity also allows the momentum expression above to be written in classical form as a mass times a velocity, i.e. as $\overrightarrow{p} = m\overrightarrow{w}$. Hence it serves one of the "type A" goals served by m' above. However, it remains an interesting but "homeless" quantity in the present literature. In other words, proper-velocity differs fundamentally from the familiar coordinate-velocity, and unlike the latter has not in textbooks been linked to a particular reference frame. After all, it uses distance measured in an inertial frame but time measured on the clocks of a moving and possibly accelerated observer. However, there is also something deeply physical about proper-velocity. Unlike coordinate-velocity, it is a synchrony-free[7,8] (i.e. local-clock only) means of quantifying motion. Moreover, A. Ungar has recently made the case[9] that proper velocities, and not coordinate velocities, make up the gyrogroup analog to the velocity group in classical physics. If so, then, is it possible that introductory students might gain deeper physical insight via its use?

The answer is yes. We show here that proper-velocity, when introduced as part of a "one-map two-clock" set of time/velocity variables, allows us to introduce relativistic momentum, time-dilation, *and* frame-invariant relativistic acceleration/force into the classroom without invoking discussion of multiple inertial frames or the abstract mathematics of Lorentz transformation (item A above). Moreover, through use of proper-velocity many relationships (including those like velocity-addition, which require multiple frames or more than one "map") are made simpler and sometimes more useful. When one simplification brings with it many others, this suggests that "item B" insights may be involved as well. The three sections to follow deal with the basic, acceleration-related, and multi-map applications of this "two-clock" approach by first developing the equations, and then discussing classroom applications.

II. A TRAVELER, ONE MAP, AND TWO CLOCKS

One may argue that a fundamental break between classical and relativistic kinematics involves the observation that time passes differently for moving observers, than it does for stationary ones. In typical texts, discussion of this fact involves discussion of separate traveler and map (e.g. primed and unprimed) inertial reference frames, perhaps including Lorentz transforms between them, even though the traveler may be accelerated and changing reference frames constantly! This is not needed. Instead, we define two time variables when describing the motion of a single object (or "traveler") with respect to a single inertial coordinate frame (or "map"). These time variables are the "map" or coordinate-time t, and the "traveler" or proper-time τ. However, only one measurement of distances will be considered, namely that associated with the inertial reference frame or "map".

It follows from above that two velocities will arise as well, namely the coordinate-velocity $\vec{v} \equiv d\vec{x}/dt$, and proper-velocity $\vec{w} \equiv d\vec{x}/d\tau$. The first velocity measures map-distance traveled per unit map time, while the latter measures map-distance traveled per unit traveler time. Each of these velocities can be calculated from the other by knowing the velocity-dependence of the "traveler's speed of map-time" $\gamma \equiv dt/d\tau$, since it is easy to see from the definitions above that:

$$\vec{w} = \gamma \vec{v}. \tag{1}$$

Because all displacements dx are defined with respect to our map frame, proper-velocity is not simply a coordinate-velocity measured with respect to a different map. However, it does have a well-defined home, in fact with many "brothers and sisters" who live there as well. This family is comprised of the velocities reported by the infinite number of moving observers who might choose to describe the motion of our traveler, with their own clock on the map of their common "home" frame of reference[10]. One might call the members of this family "non-coordinate velocities", to distinguish them from the coordinate-velocity measured by an inertial observer who stays put in the frame of the map. The cardinal rule for all such velocities is: *everyone measures displacements from the vantage point of the home frame* (e.g. on a copy of a reference-frame map in their own vehicle's glove compartment). Thus proper velocity $\vec{w}$ is that particular non-coordinate velocity which reports the rate at which a given traveler's position on the reference map changes, per unit time *on the clock of the traveler.*

A. Developing the basic equations

A number of useful relationships for the above "traveler's speed of map-time" γ, including it's familiar relationship to coordinate-velocity, follow simply from the nature of the flat spacetime metric. Their derivation is outlined in Appendix A. For students not ready for four-vectors, however, one can simply quote Einstein's prediction that spacetime is tied together so that instead of $\gamma = 1$, one has $\gamma = 1/\sqrt{1-(v/c)^2} = E/mc^2$, where E is Einstein's "relativistic energy" and c is the speed of light. By solving eqn. 1 for $w(v)$, and putting the inverted solution $v(w)$ into the expression for γ above, the following string of useful relationships follow immediately:

$$\begin{aligned}\gamma &\equiv \frac{dt}{dt_o} \\ &= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \sqrt{1+\left(\frac{w}{c}\right)^2} = \frac{E}{mc^2} = 1 + \frac{K}{mc^2}.\end{aligned} \tag{2}$$

Here of course K is the kinetic energy of motion, equal classically to $\frac{1}{2}mv^2$.

Because equation 2 allows one to relate velocities to energy, an important part of relativistic dynamics is in hand as well. Another important part of relativistic dynamics, mentioned in the introduction, takes on familiar form since momentum at any speed is

$$\vec{p} = m\vec{w} = m\gamma\vec{v}. \tag{3}$$

This relation has important scientific consequences as well. It shows that momentum like proper velocity has no upper limit, and that coordinate velocity becomes irrelevant to tracking momentum at high speeds (since for $w \gg c$, $v \simeq c$ and hence $p \propto \gamma$). All of the equations in this section are summarized for reference and comparison in Table I.

B. Basic classroom applications

One of the simplest exercises a student might perform is to show that, as proper velocity w goes to infinity, the coordinate-velocity v never gets larger than speed limit c. This can be done by simply solving equation 1 for v as a function of w. Since student intuition should argue strongly against "map-distance per unit traveler time" becoming infinite, an upper limit on coordinate-velocity v may thus from the beginning seem a very reasonable consequence. In typical introductory courses, this upper limit on coordinate-velocity is not something students are given a chance to prove for themselves. Students can also show, for themselves at this point, that *classical* kinematics follows when all speeds involved obey $v \ll c$, since this implies that $\gamma \simeq 1$ (cf. Table I).

Given these tools to describe the motion of an object with respect to single map frame, another type of relativistic problem within range is that of time dilation. From the very definition of γ as a "traveler's speed of map-time", and the velocity relations which show that $\gamma \geq 1$, it is easy for a student to see that the traveler's clock will always run slower than map time. Hence if the traveler holds a fixed speed for a finite time, one has from equation 2 that traveler time is dilated (spread out over a larger interval) relative to coordinate time, by the relation

$$\Delta t = \gamma \Delta \tau \geq \Delta \tau \tag{4}$$

Thus time-dilation problems can be addressed. This is one of several skills that this strategy can offer to students taking only introductory physics, an "item A" benefit according to the introduction. A *practical* awareness of the non-global nature of time thus does not require readiness for the abstraction of Lorentz transforms.

Convenient units for coordinate-velocity are [lightyears per map-year] or [c]. Convenient units for proper-velocity, by comparison, are [lightyears per traveler year] or [ly/tyr]. When proper-velocity reaches 1 [ly/tyr], coordinate-velocity is $\frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} \simeq 0.707$ [c]. Thus $w = 1$ [ly/tyr] is a natural dividing line between classical and relativistic regimes. In the absence of an abbreviation with mnemonic value for 1 [ly/tyr], students sometimes call it a "roddenberry" [rb], perhaps because in english this name evokes connections to "hotrodding" (high-speed), berries (minimal units for fruit), and a science fiction series which ignores the lightspeed limit to which coordinate-velocity adheres. It is also worth pointing out to students that, when measuring times in years, and distances in light years, one earth gravity of acceleration is conveniently $g \simeq 1.03$ [ly/yr^2].

We show here that the major difference between classical and two-clock relativity involves the dependence of kinetic energy K on velocity. Instead of $\frac{1}{2}mv^2$, one has $mc^2(\sqrt{1+\left(\frac{w}{c}\right)^2} - 1)$ which by Taylor expansion in $\frac{w}{c}$ goes as $\frac{1}{2}mw^2$ when $w \ll c$. Although the relativistic expression is more complicated, it is not prohibitive for introductory students, especially since they can first calculate the physically interesting "speed of map-time" γ, and then figure $K = mc^2(\gamma - 1)$. If they are given rest-energy equivalents for a number of common masses (e.g. for electrons $m_e c^2 \simeq 511keV$), this might make calculation of relativistic energies even less painful than in the classical case!

Concerning momenta, one might imagine from its definition that proper-velocity w is the important speed to a relativistic traveler trying to get somewhere on a map (say for example to Chicago) with minimum traveler time. Equation 3 shows that it is also a more interesting speed from the point of view of law enforcement officials wishing to minimize fatalities on futuristic highways where relativistic speeds are an option. Proper velocity tells us what is physically important, since is proportional to the momentum available in the collision. If we want to ask how long it will take an ambulance to get to the scene of an accident, then of course coordinate velocity may be the key.

Given that proper velocity is the most direct link to physically important quantities like traveler-time and momentum, it is not surprising that a press unfamiliar with this quantity does not attend excitedly, for example, to new settings of the "land speed record" for fastest accelerated particle. New progress changes the value of v, the only velocity they are prepared to talk about, in the 7th or 8th decimal place. The story of increasing proper velocity, thus, goes untold to a public whose imagination might be captured thereby. Thus proper-velocities for single $50GeV$ electrons in the LEP2 accelerator at CERN might be approaching $w = \gamma v = \frac{E}{mc^2} \times v \simeq \frac{50GeV}{511keV} \times c \simeq 10^5$ [lightyears per traveler year], while the educated lay public (comprised of those who have had no more than an introductory physics course) is under a vague impression that the lightspeed limit rules out major progress along these lines.

III. ACCELERATION AND FORCE WITH ONE MAP AND TWO CLOCKS.

The foregoing relations introduce, in context of a single inertial frame and without Lorentz transforms, many of the kinematical and dynamical relations of special relativity taught in introductory courses, in modern physics courses, and perhaps even in some relativity courses. In this section, we cover less familiar territory, namely the equations of relativistic acceleration. Forces if defined simply as rates of momentum-change in special relativity have no frame-invariant formulation. That is, different map frames will see different forces acting on a given traveler. Moreover, solving problems with coordinate 3-vector acceleration alone can be very messy, indeed[1].

Because of this, relativistic acceleration is seldom discussed in introductory courses. Can relativistic equations for constant acceleration, instead, be cast in familiar form? The answer is yes: a frame-invariant 3-vector acceleration, with simple integrals, arises naturally in one-map two-clock relativity. Although the development of the (3+1)D equations is tedious, we show that this acceleration bears a familiar relationship to the frame-independent rate of momentum change (i.e. the force) *felt* by an accelerated traveler.

A. Developing the acceleration equations.

By examining the frame-invariant scalar product of the acceleration 4-vector, one can show (as we do in the Appendix) that a "proper acceleration" $\overrightarrow{\alpha}$ for our traveler, which is the same to all inertial observers and thus "frame-invariant", can be written in terms of components for the classical acceleration vector $\overrightarrow{a}$ by:

$$\overrightarrow{\alpha} = \frac{\gamma^3}{\gamma_\perp}\overrightarrow{a}, \text{where } \overrightarrow{a} \equiv \frac{d^2\overrightarrow{x}}{dt^2}. \tag{5}$$

This is remarkable, given that $\overrightarrow{a}$ is so strongly frame-dependent! Here the "transverse time-speed" $\gamma_\perp$ is defined as $1/\sqrt{1-(v_\perp/c)^2}$, where $v_\perp$ is the component of coordinate velocity $\overrightarrow{v}$ perpendicular to the direction of coordinate acceleration $\overrightarrow{a}$. In this section generally, in fact, subscripts $\parallel$ and $\perp$ refer to parallel and perpendicular component-directions relative to the direction of this frame-invariant acceleration 3-vector $\overrightarrow{\alpha}$, and not (for example) relative to coordinate velocity $\overrightarrow{v}$.

Before considering integrals of the motion for constant proper acceleration $\overrightarrow{\alpha}$, let's review the classical integrals of motion for constant acceleration $\overrightarrow{a}$. These can be written as $a \simeq \Delta v_\parallel/\Delta t \simeq \frac{1}{2}\Delta(v^2)/\Delta x_\parallel$. The first of these is associated with conservation of momentum in the absence of acceleration, and the second with the work-energy theorem. These may look more familiar in the form $v_{\parallel f} \simeq v_{\parallel i} + a\Delta t$, and $v^2_{\parallel f} \simeq v^2_{\parallel i} + 2a\Delta x_\parallel$. Given that coordinate velocity has an upper limit at the speed of light, it is easy to imagine why holding coordinate acceleration constant in relativistic situations requires forces which change even from the traveler's point of view, and is not possible at all for $\Delta t > (c - v_{\parallel i})/a$.

Provided that proper time τ, proper velocity w, and time-speed γ can be used as variables, three simple integrals of the proper acceleration can be obtained by a procedure which works for integrating other non-coordinate velocity/time expressions as well[10]. The resulting integrals are summarized in compact form, like those above, as

$$\alpha = \gamma_\perp\frac{\Delta w_\parallel}{\Delta t} = c\frac{\Delta\eta_\parallel}{\Delta\tau} = \frac{c^2}{\gamma_\perp}\frac{\Delta\gamma}{\Delta x_\parallel}. \tag{6}$$

Here the integral with respect to proper time τ has been simplified by further defining the hyperbolic velocity angle or rapidity[11] $\eta_\parallel \equiv \sinh^{-1}[w_\parallel/c] = \tanh^{-1}[v_\parallel/c]$. Note that both $v_\perp$ and the "transverse time-speed" $\gamma_\perp$ are constants, and hence both proper velocity, and longitudinal momentum $p_\parallel \equiv m w_\parallel$, change at a uniform rate when proper acceleration is held constant. If motion is only in the direction of acceleration, $\gamma_\perp$ is 1, and $\Delta p/\Delta t = m\alpha$ in the classical tradition.

In classical kinematics, the rate at which traveler energy E increases with time is frame-dependent, but the rate at which momentum p increases is invariant. In special relativity, these rates (when figured with respect to proper time) relate to each other as time and space components, respectively, of the acceleration 4-vector. Both are frame-*dependent* at high speed. However, we can define proper force separately as the force *felt* by an accelerated object. We show in the Appendix that this is simply $\overrightarrow{F} \equiv m\overrightarrow{\alpha}$. That is, all accelerated objects *feel* a frame-invariant 3-vector force $\overrightarrow{F}$ in the direction of their acceleration. The magnitude of this force can be calculated from any inertial frame, by multiplying the rate of momentum change *in the acceleration direction* times $\gamma_\perp$, or by multiplying mass times the proper acceleration α. The classical relation $F \simeq dp/dt \simeq mdv/dt = md^2x/dt^2 = ma$ then becomes:

$$\begin{aligned} F &= \gamma_\perp\frac{dp_\parallel}{dt} = m\gamma_\perp\frac{dw_\parallel}{dt} = m\gamma_\perp\frac{d(\gamma v_\parallel)}{dt} \\ &= m\frac{\gamma^3}{\gamma_\perp}\frac{dv_\parallel}{dt} = m\frac{\gamma^3}{\gamma_\perp}\frac{d^2x_\parallel}{dt^2} = m\frac{\gamma^3}{\gamma_\perp}a = m\alpha \end{aligned} \tag{7}$$

Even though the rate of momentum change joins the rate of energy change in becoming frame-dependent at high speed, Newton's 2nd Law for 3-vectors thus retains a frame-invariant form.

Although they depend on the observer's inertial frame, it is instructive to write out the components of momentum and energy rate-of-change in terms of proper force magnitude F. The classical equation relating rates of momentum change to force is $d\overrightarrow{p}/dt \simeq \overrightarrow{F} \simeq ma\overrightarrow{i_\parallel}$, where $\overrightarrow{i_\parallel}$ is the unit vector in the direction of acceleration. This becomes

$$\frac{d\overrightarrow{p}}{dt} = F\left[\left(\frac{1}{\gamma_\perp}\right)\overrightarrow{i_\parallel} + \left(\frac{\gamma_\perp v_\perp}{c}\frac{v_\parallel}{c}\right)\overrightarrow{i_\perp}\right]. \tag{8}$$

Note that if there are non-zero components of velocity in directions *both* parallel and perpendicular to the direction of acceleration, then momentum changes are seen to have a component perpendicular to the acceleration direction, as well as parallel to it. These transverse momentum changes result because transverse proper velocity $w_\perp = \gamma v_\perp$ (and hence momentum $p_\perp$) changes when traveler γ changes, even though $v_\perp$ is staying constant.

As mentioned above, the rate at which traveler energy increases with time classically depends on traveler velocity through the relation $dE/dt \simeq Fv_\parallel \simeq m(\overrightarrow{a} \bullet \overrightarrow{v})$. Relativistically, this becomes

$$\frac{1}{\gamma_\perp}\frac{dE}{dt} = Fv_\parallel = m\left(\overrightarrow{\alpha} \bullet \overrightarrow{v}\right). \tag{9}$$

Hence the rate of traveler energy increase is in form very similar to that in the classical case.

Similarly, the classical relationship between work, force, and impulse can be summarized with the relation $dE/dx_{\|} \simeq \overrightarrow{F} \simeq dp_{\|}/dt$. Relativistically, this becomes

$$\frac{1}{\gamma_{\perp}} \frac{dE}{dx_{\|}} = F = \gamma_{\perp} \frac{dp_{\|}}{dt}. \tag{10}$$

Once again, save for some changes in scaling associated with the "transverse time-speed" constant $\gamma_{\perp}$, the form of the classical relationship between work, force, and impulse is preserved in the relativistic case. Since these simple connections are a result, and not the reason, for our introduction of proper time/velocity in context of a single inertial frame, we suspect that they provide insight into relations that are true both classically and relativistically, and thus are benefits of "type B" discussed in the introduction.

The development above is of course too complicated for an introductory class. However, for the case of unidirectional motion, and constant acceleration from rest, the Newtonian equations have exact relativistic analogs except for the changed functional dependence of kinetic energy on velocity. These equations are summarized in Table II.

B. Classroom applications of relativistic acceleration and force.

In order to visualize the relationships defined by equation 6, it is helpful to plot for the (1+1)D or $\gamma_{\perp} = 1$ case all velocities and times versus x in dimensionless form from a common origin on a single graph (i.e. as v/c, $\alpha\tau/c$, $w/c = \alpha t/c$, and γ versus $\alpha x/c^2$). As shown in Fig. 1, v/c is asymptotic to 1, $\alpha\tau/c$ is exponential for large arguments, $w/c = \alpha t/c$ are hyperbolic, and also tangent to a linear γ for large arguments. The equations underlying this plot, from 6 for $\gamma_{\perp} = 1$ and coordinates sharing a common origin, can be written simply as:

$$\frac{\alpha x}{c^2} + 1 = \sqrt{1 + \left(\frac{\alpha t}{c}\right)^2} = \cosh\left[\frac{\alpha\tau}{c}\right]$$
$$= \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \sqrt{1 + \left(\frac{w}{c}\right)^2}. \tag{11}$$

This universal acceleration plot, adapted to the relevant range of variables, can be used to illustrate the solution of, and possibly to graphically solve, *any* constant acceleration problem. Similar plots can be constructed for more complicated trips (e.g. accelerated twin-paradox adventures) and for the (3+1)D case as well[10].

With plots of this sort, high school students can solve relativistic acceleration problems with no equations at all! For example, consider constant acceleration from rest at $\alpha = 1$ [earth gravity] $\simeq 1$ [ly/yr^2] over a distance of 4 [lightyears]. One can read directly from Fig.1 by drawing a line up from 4 on the x-axis that $\gamma \simeq 5$, final proper-speed $w \simeq 4.8$ [ly/tyr], map-time $t \simeq 4.8$ [yr], proper-time elapsed $\tau \simeq 2.3$ [yr], and final coordinate-speed $v \simeq 1$ [c]. Problems with most initial value sets can be solved similarly, without equations, on such a plot with help from a straight edge and a bit of trial and error. Of course, the range of variables involved must be reflected in the ranges of the plot. For this reason, it may also prove helpful to replot Fig.1 on a logarithmic scale. As you can see here, in the classical limit when $w << 1$ [ly/tyr], all variables except γ (dimensionless times and velocities alike) *take on the same value* as a function of distance traveled from rest!

For a numerical example, imagine trying to predict how far one might travel by accelerating at one earth gravity for a fixed traveler-time, and then turning your thrusters around and decelerating for the same traveler-time until you are once more at rest in your starting or "map" frame. To be specific, consider the 14.2 proper-year first half of such a trip all the way to the Andromeda galaxy[12], one of the most distant (and largest) objects visible to the naked eye. From equation 6, the maximum (final) rapidity is simply $\eta_{\|} = \alpha\tau/c = 14.7$. Hence the final proper velocity is $w = \sinh(\alpha\tau/c) = 1.2 \times 10^6 ly/tyr$. From equation 2 this means that $\gamma = \sqrt{1 + (w/c)^2} = 1.2 \times 10^6$, and the coordinate velocity $v = w/\sqrt{1 + (w/c)^2} = 0.99999999999963 ly/yr$. Going back to equation 6, this means that coordinate time elapsed is $t = w/\alpha c = 1.1 \times 10^6 years$, and distance traveled $x = (\gamma - 1)c^2/\alpha = 1.1 \times 10^6 ly$. Few might imagine, from typical intro-physics treatments of relativity, that one could travel over a million lightyears in less than 15 years on the traveler's clock!

From equation 5, the coordinate acceleration falls from $1 gee$ at the start of the leg to $a = \alpha/\gamma^3 = 6 \times 10^{-19} gee$ at maximum speed. The forces, energies, and momenta of course depend on the spacecraft's mass. At any given point along the trajectory from the equations above, F is of course just $m\alpha$, dE/dx is $\gamma_{\perp}F = F$, dp/dt is $F/\gamma_{\perp} = F$, and dE/dt is $\gamma_{\perp}Fv_{\|} = Fv$. Note that all except the last of these are constant if mass is constant, albeit dependent on the reference frame chosen. However, the 4-vector components $dp/d\tau$ and $dE/d\tau$ are not constant at all, showing in another way the pervasive frame-dependences mentioned above.

The foregoing solution may seem routine, as well it should be. It is not. Note that it was implemented using distances measured (and concepts defined) in context of a single map frame. Moreover, the 3-vector forces and accelerations used and calculated have frame-invariant components, i.e. those particular parameters are correct in context of all inertial frames.

The mass of the ship in the problem above may vary with time. For example, if the spacecraft is propelled by ejecting particles at velocity u opposite to the acceleration direction, the force felt in the frame of the traveler will be simply $m\alpha = -u dm/d\tau$. Hence in terms of traveler time the mass obeys $m = m_o \exp[-\alpha\tau/u]$. In terms of coordinate time, the differential equation becomes $m\alpha = -u\gamma dm/dt$. This can be solved to get the solution derived with significantly more trouble in the reference above[12].

IV. PROBLEMS INVOLVING MORE THAN ONE MAP.

The foregoing sections treat calculations made possible, and analogies with classical forms which result, if one introduces the proper time/velocity variables in the context of a single map frame. What happens when multiple map-frames are required? In particular, are the Lorentz transform and other multi-map relations similarly simplified or extended? The answer is yes, although our insights in this area are limited since the focus of this paper is introductory physics, and not special relativity.

A. Development of multi-map equations

The Lorentz transform itself is simplified with the help of proper velocity, in that it can be written in the symmetric matrix form:

$$\begin{bmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{pmatrix} \gamma & \pm\frac{w}{c} & 0 & 0 \\ \pm\frac{w}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}. \tag{12}$$

This seems to be an improvement over the asymmetric equations normally used, but of course requires a bit of matrix and 4-vector notation that your students may not be ready to exploit.

The expression for length contraction, namely $L = L_o/\gamma$, is not changed at all. The developments above do suggest that the concept of proper length L_o, as the length of a yardstick in the frame in which it is at rest, may have broader use as well. The relativistic Doppler effect expression, given as $f = f_o\sqrt{\{1+(v/c)\}/\{1-(v/c)\}}$ in terms of coordinate velocity, also simplifies to $f = f_o/\{\gamma - (w/c)\}$. The classical expression for the Doppler effected frequency of a wave of velocity v_{wave} from a moving source of frequency f_o is, for comparison, $f = f_o/\{1-(v/v_{wave})\}$.

The most noticeable effect of proper velocity, on the multi-map relationships considered here, involves simplification and symmetrization of the velocity addition rule. The rule for adding coordinate velocities $\overrightarrow{v}'$ and $\overrightarrow{v}$ to get relative coordinate velocity $\overrightarrow{v}''$, namely $v''_\parallel = (v' + v_\parallel)/(1 + v_\parallel v'/c^2)$ and $v''_\perp \neq v_\perp$ with subscripts referring to component orientation with respect to the direction of $\overrightarrow{v}'$, is inherently complicated. Moreover, for high speed calculations, the answer is usually uninteresting since large coordinate velocities always add up to something very near to c. By comparison, if one adds proper velocities $\overrightarrow{w}' = \gamma'\overrightarrow{v}'$ and $\overrightarrow{w} = \gamma\overrightarrow{v}$ to get relative proper velocity $\overrightarrow{w}''$, one finds simply that the coordinate velocity factors add while the γ-factors multiply, i.e.

$$w''_\parallel = \gamma'\gamma\left(v' + v_\parallel\right), \text{with } w'_\perp = w_\perp. \tag{13}$$

Note that the components transverse to the direction of $\overrightarrow{v}'$ are unchanged. These equations are summarized for the unidirectional motion case in Table III.

B. Classroom applications involving more than one map-frame.

Physically more interesting questions can be answered with equation 13 than with the coordinate velocity addition rule commonly given to students. For example, one might ask what the speed record is for *relative* proper velocity between two objects accelerated by man. For the world record in this particle-based demolition derby, consider colliding two beams from an accelerator able to produce particles of known energy for impact onto a stationary target. From Table I for colliding $50GeV$ electrons in the LEP2 accelerator at CERN, γ and γ' are $E/mc^2 \simeq 50GeV/511keV \simeq 10^5$, v and v' are essentially c, and w and w' are hence 10^5c. Upon collision, equation 13 tells us that the relative proper speed w'' is $(10^5)^2(c+c) = 2\times10^{10}c$. Investment in a collider thus buys a factor of $2\gamma = 2\times10^5$ increase in the momentum (and energy) of collision. Compared to the cost of building a $10PeV$ accelerator for the equivalent effect on a stationary target, the collider is a bargain indeed!

V. CONCLUSIONS.

We show in this paper that a one-map two-clock approach, using both proper and coordinate velocities, lets students tackle time dilation as well as momentum and energy conservation problems without having to first master concepts which arise when considering more than one inertial frame (like Lorentz transforms, length contraction, and frame-dependent simultaneity). The cardinal rule to follow when doing this is simple: All distances must be defined with respect to a “map” drawn from the vantage point of a single inertial reference frame.

We show further that a *frame-invariant* proper acceleration 3-vector has three simple integrals of the motion in terms of these variables. Hence students can speak of the proper acceleration and force 3-vectors for an object in map-independent terms, and solve relativistic constant acceleration problems much as they now do for non-relativistic problems in introductory courses.

We have provided some examples of the use of these equations for high school and college introductory physics classes, as well as summaries of equations for the simple unidirectional motion case (Tables I, II, and III). In the process, one can see that the approach does more than "superficially preserve classical forms". Not just one, but many, classical expressions take on relativistic form with only minor change. In addition, interesting physics is accessible to students more quickly with the equations that result. The relativistic addition rule for proper velocities is a special case of the latter in point. Hence we argue that the trend in the pedagogical literature, away from relativistic masses and toward use of proper time and velocity in combination, may be a robust one which provides: (B) deeper insight, as well as (A) more value from lessons first-taught.

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APPENDIX A: THE 4-VECTOR PERSPECTIVE

This appendix provides a more elegant view of matters discussed in the body of this paper by using space-time 4-vectors not used there, along with some promised derivations. We postulate first that: **(i)** displacements between events in space and time may be described by a displacement 4-vector $\mathbf{X}$ for which the time–component may be put into distance-units by multiplying by the speed of light c; **(ii)** subtracting the sum of squares of space-related components of any 4-vector from the time component squared yields a scalar "dot-product" which is *frame-invariant*, i.e. which has a value which is the same for all inertial observers; and **(iii)** translational momentum and energy, two physical quantities which are conserved in the absence of external intervention, are components of the momentum-energy 4-vector $\mathbf{P} \equiv m\frac{d\mathbf{X}}{d\tau}$, where m is the object's rest mass and τ is the frame-invariant displacement in time-units along its trajectory.

From above, the *4-vector displacement* between two events in space-time is described in terms of the position and time coordinate values for those two events, and can be written as:

$$\Delta\mathbf{X} \equiv \begin{bmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}. \tag{A1}$$

Here the usual Δ-notation is used to represent the value of final minus initial. The dot-product of the displacement 4-vector is defined as the square of the frame-invariant proper-time interval between those two events. In other words,

$$\begin{aligned}(c\Delta\tau)^2 &\equiv \Delta\mathbf{X} \bullet \Delta\mathbf{X} \\ &= (c\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2).\end{aligned} \tag{A2}$$

Since this dot-product can be positive or negative, proper time intervals can be real (time-like) or imaginary (space-like). It is easy to rearrange this equation for the case when the displacement is infinitesimal, to confirm the first two equalities in equation 2 via:

$$\gamma \equiv \frac{dt}{d\tau} = \sqrt{1 + \left(\frac{dx}{d\tau}\right)^2} = \frac{1}{\sqrt{1 - \left(\frac{dx}{dt}\right)^2}}. \tag{A3}$$

The *momentum-energy 4-vector*, as mentioned above, is then written using γ and the components of proper velocity $\vec{w} \equiv \frac{d\vec{x}}{d\tau}$ as:

$$\mathbf{P} \equiv m\mathbf{U} = m \begin{bmatrix} c\gamma \\ w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{bmatrix}. \tag{A4}$$

Here we've also taken the liberty to use a *velocity 4-vector* $\mathbf{U} \equiv d\mathbf{X}/d\tau$. The equality in equation 2 between γ and E/mc^2 follows immediately. The frame-invariant dot-product of this 4-vector, times c squared, yields the familiar relativistic relation between total energy E, momentum p, and frame-invariant rest mass-energy mc^2:

$$c^2\mathbf{P} \bullet \mathbf{P} = \left(mc^2\right)^2 = E^2 - (cp)^2. \tag{A5}$$

If we define kinetic energy as the difference between rest mass-energy and total energy using $K \equiv E - mc^2$, then the last equality in equation 2 follows as well. Another useful relation which follows is the relation between infinitesimal uncertainties, namely $\frac{dE}{dp} = \frac{dx}{dt}$.

Lastly, the *force-power 4-vector* may be defined as the proper time derivative of the momentum-energy 4-vector, i.e.:

$$\mathbf{F} \equiv \frac{d\mathbf{P}}{d\tau} \equiv m\mathbf{A} = m \begin{bmatrix} c\frac{d\gamma}{d\tau} \\ \frac{dw_x}{d\tau} \\ \frac{dw_y}{d\tau} \\ \frac{dw_z}{d\tau} \end{bmatrix} = \begin{bmatrix} \frac{1}{c}\frac{dE}{d\tau} \\ \frac{dp_x}{d\tau} \\ \frac{dp_y}{d\tau} \\ \frac{dp_z}{d\tau} \end{bmatrix}. \tag{A6}$$

Here we've taken the liberty to define *acceleration 4-vector* $\mathbf{A} \equiv d^2\mathbf{X}/d\tau^2$ as well.

The dot-product of the force-power 4-vector is always negative. It may therefore be used to define the frame-invariant proper acceleration α, by writing:

$$\mathbf{F} \bullet \mathbf{F} \equiv -(m\alpha)^2 = \left(\frac{1}{c}\frac{dE}{d\tau}\right)^2 - \left(\frac{dp}{d\tau}\right)^2. \tag{A7}$$

We still must show that this frame-invariant proper acceleration has the magnitude specified in the text (eqn. 5). To relate proper acceleration α to coordinate acceleration $\overrightarrow{a} \equiv \frac{d\overrightarrow{v}}{dt} \equiv \frac{d^2\overrightarrow{x}}{dt^2}$, note first that $c\frac{d\gamma}{d\tau} = \gamma^4 \frac{v_\parallel}{c} a$, that $\frac{dw_\parallel}{d\tau} = \frac{\gamma^4}{\gamma_\perp^2} a$, and that $\frac{dw_\perp}{d\tau} = \gamma^3 \frac{v_\perp}{c} \frac{v_\parallel}{c} a$. Putting these results into the dot-product expression for the fourth term in A6 and simplifying yields $\alpha^2 = \frac{\gamma^6}{\gamma_\perp^2} a^2$ as required.

As mentioned in the text, power is classically frame-dependent, but frame-dependence for the components of momentum change only asserts itself at high speed. This is best illustrated by writing out the force 4-vector components for a trajectory with constant proper acceleration, in terms of *frame-invariant* proper time/acceleration variables τ and α. If we consider separately the momentum-change components parallel and perpendicular to the unchanging and frame-independent acceleration 3-vector $\overrightarrow{\alpha}$, one gets

$$\mathbf{F} = \begin{bmatrix} \frac{1}{c}\frac{dE}{d\tau} \\ \frac{dp_\parallel}{d\tau} \\ \frac{dp_\perp}{d\tau} \\ 0 \end{bmatrix} = m\alpha \begin{bmatrix} \gamma_\perp \sinh\left[\frac{\alpha\tau}{c} + \eta_o\right] \\ \cosh\left[\frac{\alpha\tau}{c} + \eta_o\right] \\ \gamma_\perp \frac{v_\perp}{c} \sinh\left[\frac{\alpha\tau}{c} + \eta_o\right] \\ 0 \end{bmatrix}, \tag{A8}$$

where η_o is simply the initial value for $\eta_\parallel \equiv \sinh^{-1}[\frac{w_\parallel}{c}]$.

The force responsible for motion, as distinct from the frame-dependent rates of momentum change described above, is that seen by the accelerated object itself. As equation A8 shows for $\tau, v_\perp$ and η_o set to zero, this is nothing more than $\overrightarrow{F} \equiv m\overrightarrow{\alpha}$. Thus some utility for the rapidity/proper time integral of the equations of constant proper acceleration (3rd term in eqn. 6) is illustrated as well.

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Equation\ Version:	classical (c$\rightarrow\infty$)	two-clock relativity
speed of map-time	$\gamma \equiv \frac{dt}{d\tau} \simeq 1$	$\gamma \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \sqrt{1+\left(\frac{w}{c}\right)^2}$
time dilation	none	$\Delta t = \gamma \Delta\tau$
coordinate velocity	$\overrightarrow{v} \equiv \frac{d\overrightarrow{x}}{dt}$	$\overrightarrow{v} \equiv \frac{d\overrightarrow{x}}{dt} = \frac{\overrightarrow{w}}{\sqrt{1+\left(\frac{w}{c}\right)^2}}$
proper velocity	same as coordinate	$\overrightarrow{w} \equiv \frac{d\overrightarrow{x}}{d\tau} = \gamma\overrightarrow{v}$
momentum	$\overrightarrow{p} \simeq m\overrightarrow{v}$	$\overrightarrow{p} = m\overrightarrow{w}$
kinetic energy	$K \simeq \frac{1}{2}mv^2 \simeq \frac{p^2}{2m}$	$K = mc^2(\gamma-1) \equiv E - mc^2$
total energy $mc^2 + K$	not considered	$E = \gamma mc^2 = \sqrt{(pc)^2 + (mc^2)^2}$

TABLE I. Equations involving velocities, times, momentum and energy, in classical and two-clock relativistic form.

Equation\ Version:	classical (c$\rightarrow\infty$)	two-clock relativity
coordinate acceleration	$a \equiv \frac{dv}{dt}$	$a \equiv \frac{dv}{dt}$
"felt" or proper acceleration	same as coordinate	$\alpha = \frac{dw}{dt} = \gamma^3 a$
momentum integral	$\frac{p}{m} \simeq at \simeq v$	$\frac{p}{m} = \alpha t = w$
work-energy integral	$\frac{K}{m} \simeq ax \simeq \frac{1}{2}v^2$	$\frac{K}{m} = \alpha x = c^2(\gamma-1)$
proper-time integral	same as momentum	$\alpha\tau = c\sinh^{-1}\left(\frac{w}{c}\right)$
force, work & impulse	$F \simeq ma \simeq \frac{dE}{dx} = \frac{dp}{dt}$	$F = m\alpha = \frac{dE}{dx} = \frac{dp}{dt}$

TABLE II. Unidirectional motion equations involving constant acceleration from rest, and "2nd law" dynamics, in classical and two-clock relativistic form.

Equation\ Version:	classical (c$\rightarrow\infty$)	two-clock relativity
frame transformation	$x' \simeq x \pm vt;\ t' \simeq t$	$x' = \gamma x \pm wt;\ t' = \gamma t \pm wx/c^2$
length contraction	none	$L = L_o/\gamma$
moving-source Doppler-shift	$f \simeq \frac{f_{source}}{1 \pm v/v_{wave}}$ (any wave)	$f = \frac{f_{source}}{\gamma \pm w/c}$ (light)
velocity addition	$v_{ac} \simeq v_{ab} + v_{bc}$	$w_{ac} \simeq \gamma_{ab}\gamma_{bc}(v_{ab} + v_{bc})$

TABLE III. Unidirectional motion equations involving distances measured using more than one map-frame, in classical and two-clock relativistic form.

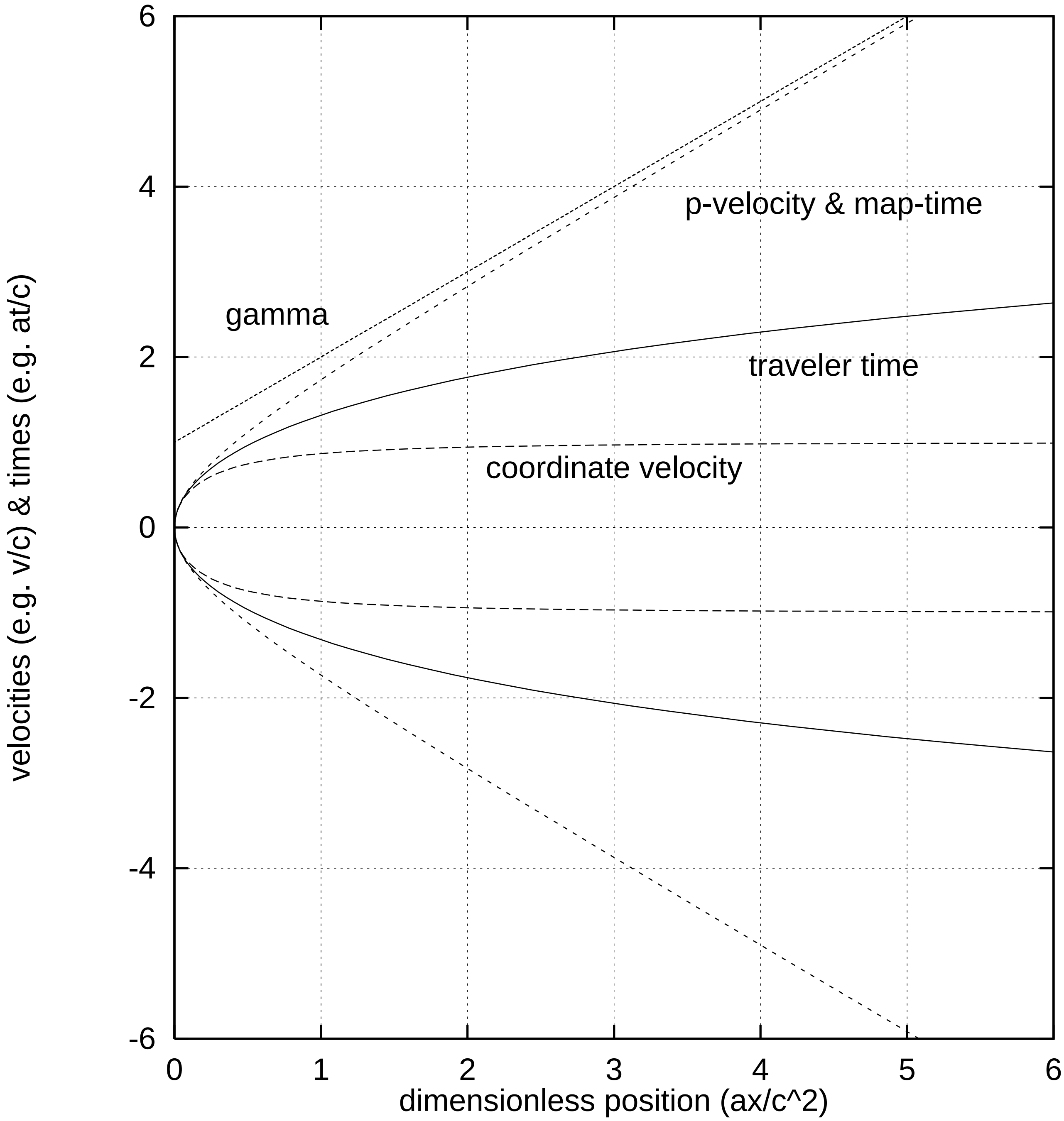

FIG. 1. The variables involved in (1+1)D constant acceleration