The proper-force 3-vector

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The distinction between proper (i.e. cell-phone detectable) and geometric (i.e. connection-coefficient) forces allows one to use Newton's 3-vector laws in accelerated frames and curved spacetime. Here we show how this is assisted by use of quantities that are either (i) frame-invariant or (ii) synchrony-free i.e. do not rely on extended-networks of synchronized-clocks. The acceleration four-vector's invariant magnitude, and quantities that build on the metric-equation's book-keeper frame to define simultaneity, point the way to more robust student understanding at *both* low and high speeds. In the process, we gain a simple (3+1)D flat-space work-energy theorem using the proper-acceleration 3-vector $\vec{\alpha}$ (net proper-force per unit mass), whose integrals of the motion simplify with a hyperbolic velocity angle (rapidity) written as $\sqrt{2/(\gamma_o+1)}\alpha\tau/c$, where c is lightspeed and τ is traveler-time from "turnaround" when the Lorentz-factor is γ_o.

CONTENTS

I. INTRODUCTION

Relativists have long expressed unhappiness with coordinate-acceleration and coordinate-force (for good reason[1,2]), but have also pointed out that general-relativity makes a case for the local-validity of Newton's laws in all frames[3–5] provided that we consider geometric (curved-frame or "connection-coefficient") forces as well as proper-forces whenever we find ourselves in a non-"free-float" trajectory[6]. The distinction has everday relevance to intro-physics students, because texts often introduce gravity as a real force but inertial forces as fake even though their cell-phones detect neither for the same reason: Accelerometers only detect proper-forces, while gravity and inertial (e.g. centrifugal) forces are geometric.

In this paper we explore an approach to accelerated motion designed to be: (i) the most frame-independent, and (ii) the least in need of synchronized-clock arrays. These latter might be difficult to come by on accelerated platforms and in curved spacetime.

The first proper-time derivative of an accelerated traveler's 4-vector position has lightspeed c as its invariant magnitude. Here we simply define simultaneity using bookkeeper coordinates and then examine the second proper-time derivative of position, as seen from the proper reference-frame[3] of that accelerated traveler.

In the process we show: (a) that the distinction between proper and geometric forces is already quite useful for introductory physics, (b) that via the metric equation a lot can be done with only a single extended map-frame of yardsticks and synchronized clocks, and (c) that the traveler's view of anyspeed-acceleration is less frame-variant than the map perspective. We also exploit the frame-invariance of proper-force in an empirical observation exercise on the electrostatic origin of magnetism, which provides some visceral experience with length-contraction at the same time.

II. FRAME DEPENDENCE & SYNCHRONY

The value of frame-independence in the modeling of relativistic-motion and curved-spacetime goes without saying. The frame-invariance of lightspeed c (the magnitude of the **velocity 4-vector** $U^\lambda \equiv dX^\lambda/d\tau$) has been central to our understanding of spacetime from the beginning[7]. Proper-time (the magnitude of the **displacement 4-vector** X^λ) is finding increasing use by introductory text authors as we speak.

The Lorentz-transform view of proper-time, of course, is that it is time-passing on the synchronized clocks of a tangent but co-moving free-float-frame in flat spacetime. The metric equation's view of proper-time is simpler but more general, i.e. as a quantity measured on a single clock under any conditions i.e. accelerated or not, in curved space-time or not.

Proper-time is frame-invariant in the sense that its value may be agreed upon using *any* general-relativistic book-keeper coordinates that we choose. These book-

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TABLE I. Accelerated-motion definitions in flat (3+1)D spacetime. Note that acceleration/force magnitudes are spacelike, while the others are timelike along a traveler's worldline, and that we've defined x and y as spatial coordinates $\parallel$ and $\perp$ to the direction of proper-acceleration 3-vector $\vec{\alpha}$.

4-vector	magnitude	time-components	$\parallel$ to spatial 3-vector $\vec{\alpha}$	$\perp$ to spatial 3-vector $\vec{\alpha}$
power/force	$\Sigma F_o \equiv \frac{dp_o}{d\tau}$	$\frac{\mathrm{P}}{c} = \left(\frac{1}{c}\right)\frac{dE}{d\tau}$	$F_{\parallel} \equiv \frac{dp_{\parallel}}{d\tau}$	$F_{\perp} \equiv \frac{dp_{\perp}}{d\tau}$
acceleration	$\alpha = \frac{\Sigma F_o}{m}$	$c\frac{d\gamma}{d\tau} = \frac{\mathrm{P}}{mc} = \gamma\frac{P}{mc}$	$\frac{dw_{\parallel}}{d\tau} = \frac{F_{\parallel}}{m} = \gamma\frac{f_{\parallel}}{m}$	$\frac{dw_{\perp}}{d\tau} = \frac{F_{\perp}}{m} = \gamma\frac{f_{\perp}}{m}$
energy/momentum	mc	$\frac{E}{c} = \gamma mc$	$p_{\parallel} = mw_{\parallel}$	$p_{\perp} = mw_{\perp}$
velocity	c	$c\gamma \equiv c\frac{dt}{d\tau} = \frac{E}{mc}$	$w_{\parallel} \equiv \frac{dx}{d\tau} = \frac{p_{\parallel}}{d\tau} = \gamma v_{\parallel}$	$w_{\perp} \equiv \frac{dx}{d\tau} = \frac{p_{\perp}}{d\tau} = \gamma v_{\perp}$
coordinate	$c\tau$	ct	x	y

TABLE II. Relationship between variables: Here τ is traveler-time elapsed from "turnaround" (when $\gamma \equiv \gamma_o$) for as long as proper acceleration $\vec{\alpha}$ doesn't change, and $\gamma_{\pm} \equiv \sqrt{(\gamma_o \pm 1)/2}$. The right arrow $\rightarrow$ denotes the non-relativistic limit.

4-vector	invariant	time-components/c	$\parallel$ to spatial 3-vector $\vec{\alpha}$	$\perp$ to spatial 3-vector $\vec{\alpha}$
accel.	α	$\frac{d\gamma}{d\tau} = \frac{\alpha}{c}\gamma_+ \sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \left(\frac{\alpha}{c}\right)^2\tau$	$\frac{dw_{\parallel}}{d\tau} = \alpha\cosh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \alpha$	$\frac{dw_{\perp}}{d\tau} = \alpha\gamma_- \sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow 0$
velocity	c	$\gamma = \gamma_-^2 + \gamma_+^2 \cosh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow 1$	$w_{\parallel} = c\gamma_+ \sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \alpha\tau$	$w_{\perp} = c\gamma_+\gamma_-(\cosh\left[\frac{\alpha\tau}{c\gamma_+}\right] + 1) \rightarrow v_{\perp}$
coord.	τ	$t = \gamma_-^2\tau + \gamma_+^3\frac{c}{\alpha}\sinh\left[\frac{\alpha\tau}{c\gamma_+}\right] \rightarrow \tau$	$x = \gamma_+^2\frac{2c^2}{\alpha}\sinh\left[\frac{\alpha\tau}{2c\gamma_+}\right]^2 \rightarrow \frac{1}{2}\alpha\tau^2$	$y = \gamma_+\gamma_-(c\tau + \frac{c}{\alpha}\sinh\left[\frac{\alpha\tau}{c\gamma_+}\right]) \rightarrow v_{\perp}\tau$

keeper coordinates are alone used to define extended simultaneity (i.e. the global place and time of events), while the frame-invariance of proper-time drastically improves the transformation-properties of quantities differentiated with respect to it. The proper-velocity 3-vector $\vec{w} \equiv d\vec{x}/d\tau$ (which unlike coordinate velocity $\vec{v} \equiv d\vec{x}/dt$ adds vectorially with appropriate rescaling of the "out-of-frame" component) and the proper-acceleration 3-vector (discussed here) are cases in point.

The topic of this paper is in particular the frame-invariant magnitude of the **acceleration 4-vector**, in standard notation[3]:

$$A^{\lambda} := \frac{DU^{\lambda}}{d\tau} = \frac{dU^{\lambda}}{d\tau} + \Gamma^{\lambda}{}_{\mu\nu}U^{\mu}U^{\nu} \tag{1}$$

and uses for this vector's components (as power/force) when they are multiplied by frame-invariant rest-mass m. Here free-float or geodesic trajectories have $A^{\lambda} = 0$, so that we can think of coordinate acceleration $dU^{\lambda}/d\tau$ as a sum of proper and geometric terms, the latter depending on local space-time curvature through the 64-component affine-connection $\Gamma^{\lambda}_{\mu\nu}$ which gives rise to "apparent" forces in accelerated coordinate-systems and curved space-time. As usual greek indices run from 0 (time-component) to 3 (space-components) and obey the Einstein summation convention when repeated in a product. Because this proper-acceleration four-vector becomes purely space-like in a frame instantaneously-comoving with our traveler, its physical interpretation is simply the proper-force/mass felt to be "pressing on" our traveler, as well as the 3-vector proper-acceleration[8–10] $\vec{\alpha}$ seen by free-float observers in the co-moving frame.

In addition to a preference here for frame-invariance, the concept of simultaneity is a messy one in accelerated frames (e.g. using radar-time methods[11]) as well as in curved spacetime. Hence we take a "metric-first" approach to kinematics here by choosing a single "book-keeper" coordinate-system in terms of which both "map-time" t and "map-position" $\vec{x}$ are measured. Simultaneity will be defined in terms of synchronized (but not always local e.g. in the case of Schwarzschild "far-time") clocks in this book-keeper frame.

In addition purely space-like vectors, along with frame-invariants, may be described as "synchrony-free" to use a word employed by William Shurcliff when discussing proper-velocity[12,13] $\vec{w} \equiv d\vec{x}/d\tau = \vec{p}/m$. These are quantities whose operational-definition does not require an extended network of synchronized-clocks, something of limited availability around gravitational-objects (like earth), and impossible to find on platforms (like spaceships) undergoing accelerated motion. The time-like energy of a moving object via its dependence on the Lorentz-factor $\gamma \equiv dt/d\tau$ is (like "mixed objects" such as coordinate-velocity $\vec{v} \equiv d\vec{x}/dt$) not synchrony-free, because it requires map-time t data from clocks at multiple locations.

The "traveler's point of view" that we argue offers the most direct way to communicate about an accelerated traveler is the frame that Misner, Thorne and Wheeler[3] refer to as "the proper reference frame of an accelerated traveler". One can always convert these to expressions written in terms of bookkeeper variables like map-time t and coordinate-velocity $\vec{v}$, but we show here that the algorithmically-simplest way to describe the effects of the local space-time metric on motion (following the critera above) involves the parameterization described here.

III. LOW SPEED APPLICATIONS

For applications at low speed, telling students about proper-forces as distinct from geometric-forces (that act on every ounce of a object's being) is a good start in

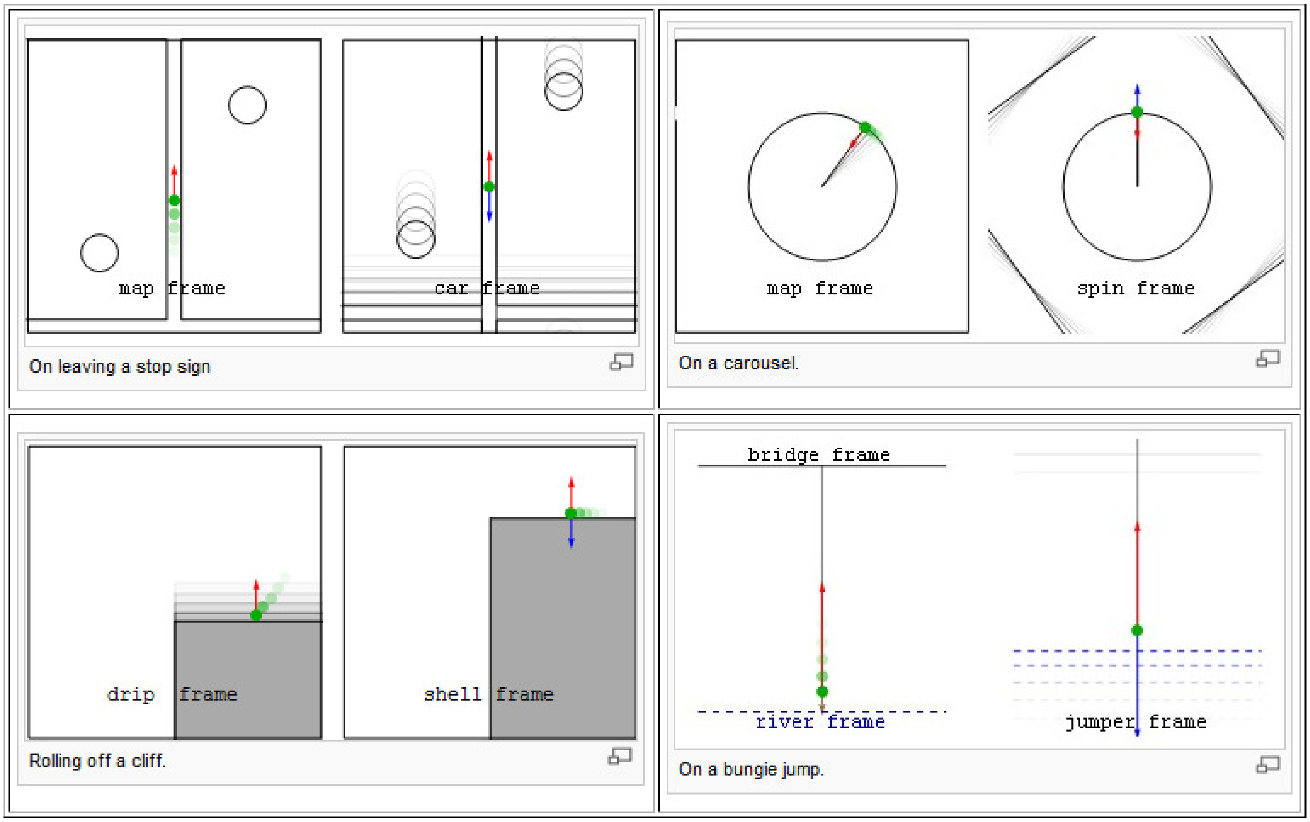

FIG. 1. Two views of proper (red) and geometric (dark blue or brown) forces in some everyday settings.

preparing them for the value of Newton's laws in both free-float and accelerated frames. The simple example (a) of a car leaving a stop-sign is illustrated by the screen capture from a wikimedia commons animation in the top-left of Fig. 1, which shows the red proper-force seen by observers in both frames as pressing on the driver's back while the car accelerates. This of course is canceled *only in the car frame* by a geometric force which (like gravity) acts on every ounce of the driver's being. Animation screen captures are also provided in that figure which show proper and geometric forces from two perspectives in the cases of (b) carousel motion, (c) the process of rolling off a cliff, and (d) the later stages of a bungie jump. The screen capture in Fig. 2 from the "real-time" animation of a 50[m] diameter rotating-wheel space-habitat with 1 "gee" of artificial gravity at its perimeter is similarly instructive.

We also recommend telling intro-physics students that time itself is dependent on a given clock's location and state of motion, with the "speed of map-time" relative to a traveler's clock (i.e. $dt/d\tau$) an important clue to the traveling-clock's energy (potential and/or kinetic). These things may be done at the outset, followed by the assertion that introductory physics texts by default refer to map-time (t) since traveler-time (τ) differences at low speed are negligible, and they traditionally treat gravity as another proper-force even though we now know that it too is a geometric-force, caused not by a traveler's motion but by gravity's curvature of space-time around massive objects. Traditional treatments often further focus only on application of Newton's laws from "inertial-frame" perspectives, in which case geometric-forces (other than gravity) can be ignored. With these minor "metric-first" changes to the introduction, traditional introductory physics treatments remain perfectly self-consistent and intact.

IV. BRINGING IN THE METRIC

In order for teachers to feel grounded when addressing introductory issues in context of an intimidating Riemann-geometry framework, it is crucial that the consequences of their assumptions be easy to *for them* to verify. Thankfully the metric-equation, unlike Lorentz transforms, requires only one bookkeeper frame whose time-variable may (or may not) be possible to associate with time's passage on clocks synchronized across a meaningful region of spacetime.

Our first step, namely choosing the metric parameterization to describe a specific problem, is especially important because it defines both the meaning of measurments and our (perhaps implicit) definition of simultaneity. This is good news for introductory teachers, since *its bad enough to be talking about different times on different clocks, without having to also be juggling multiple definitions of simultaneity.*

For general relativity applications in a world where time is measured on watches, and distances are measured with yardsticks, whenever possible we will seek metric parameterizations whose time-variable corresponds to clocks that can be synchronized. We therefore follow Newton in flat-space settings by choosing a set of free-float (e.g. inertial or un-accelerated) frame variables like coordinate-time t and coordinate-position $\vec{x}$ to describe accelerated motion.

As teachers, once we have a metric and a corresponding definition of what simultaneity means, we are back on familiar territory. The caveat is that frame-independence may be attributed only to four-vector magnitudes, and no longer to time-intervals, distances, or rates of momentum-change. For the flat-space (1+1)D case, for instance, the proper time-interval $\delta\tau$ and derivatives with respect to τ yield the following frame-invariant magnitudes:

$$(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2, \tag{2}$$

with the lightspeed constant c

$$c^2 = \left(c\frac{\delta t}{\delta\tau}\right)^2 - \left(\frac{\delta x}{\delta\tau}\right)^2, \tag{3}$$

and proper-acceleration α:

$$-\alpha^2 = \left(c\frac{\delta^2 t}{\delta\tau^2}\right)^2 - \left(\frac{\delta^2 x}{\delta\tau^2}\right)^2. \tag{4}$$

Given this, the challenge of finding the integrals of the motion e.g. for constant acceleration is much like that challenge of showing that $x = \frac{1}{2}at^2$ via the same derivative relations, but using Newton's assumptions that coordinate-intervals and coordinate-acceleration $a \equiv \frac{\delta^2 x}{\delta t^2}$ are frame-invariant. Simple-form versions of the metric-based integrals are tabulated in context of the discussions to follow.

V. ANY SPEED APPLICATIONS

Table I defines notation for describing accelerated motion in (3+1)D flat spacetime. Table II shows the instantaneous relationship between these varables (also at low speed), as parameterized by the "traveler-time τ and Lorentz-factor γ_o from turnaround" were the instantaneous proper acceleration to remain constant (cf. Appendix A). In both tables, only values in the "time-components" column rely on synchrony between map-frame clocks at more that one location. Values in the spatial-coordinate columns to the right are synchrony-free, while values in the column to the left are frame-invariant as well.

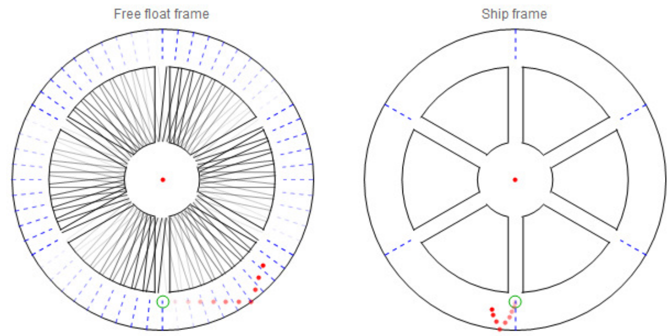

FIG. 2. Free-float and ship frame views of a pentagonal dropped-ball trajectory in a rotating-wheel space habitat, to illustrate the non-contact nature of the "cell-phone undetectable" centrifugal force.

Of course a map-frame observer's measurements of map-position as a function of map-time (along with deliverables like inferred coordinate-forces) will be parameterized in terms of synchrony-dependent map-time instead of frame-invariant traveler-time. Although map-frame observers can calculate synchrony-free quantities like momentum and proper-velocity in terms of synchrony-dependent parameters, it will take extra steps going to there from what they measure, and perhaps also going from there to what they want to infer.

If on the other hand the traveler measures their "felt proper-acceleration", as well as the rates at which they pass "map-landmarks" on their route, the equations to everything else are simpler and organically related as shown in Table II. Plus, everything that the traveler measures and reports on (except for elements in the time-component column of the table) will either be synchrony-free or frame-invariant.

The connection between the traveler control-parameters and Table II is reinforced if we imagine long-distance travel in a spacecraft with traveler control over thust (i.e. proper-force) magnitude and direction. The table connects proper-acceleration's magnitude and direction to instantaneous values of "proper-time from turnaround" and $v_\perp$, which in turn are related via the same table to navigational objectives (like the x and y values for the turnaround-point itself).

Although variable-rearrangement is complicated relative to the low-speed case via "gamma-factor" coupling between directions, a wide range of puzzles involving high-speed navigation in free-space may be addressed with this table. Of most interest perhaps to beginning students are of course the possibilities that relativity opens up for constant proper-acceleration (e.g. 1 "gee") round-trips between distant locations. Not only are these equations even simpler than the (3+1)D case, but the real limiting factor (namely the payload to launch-mass ratio) is quite simple to calculate as well.

A practical classroom application of the frame-independence of proper-force in this context involves an

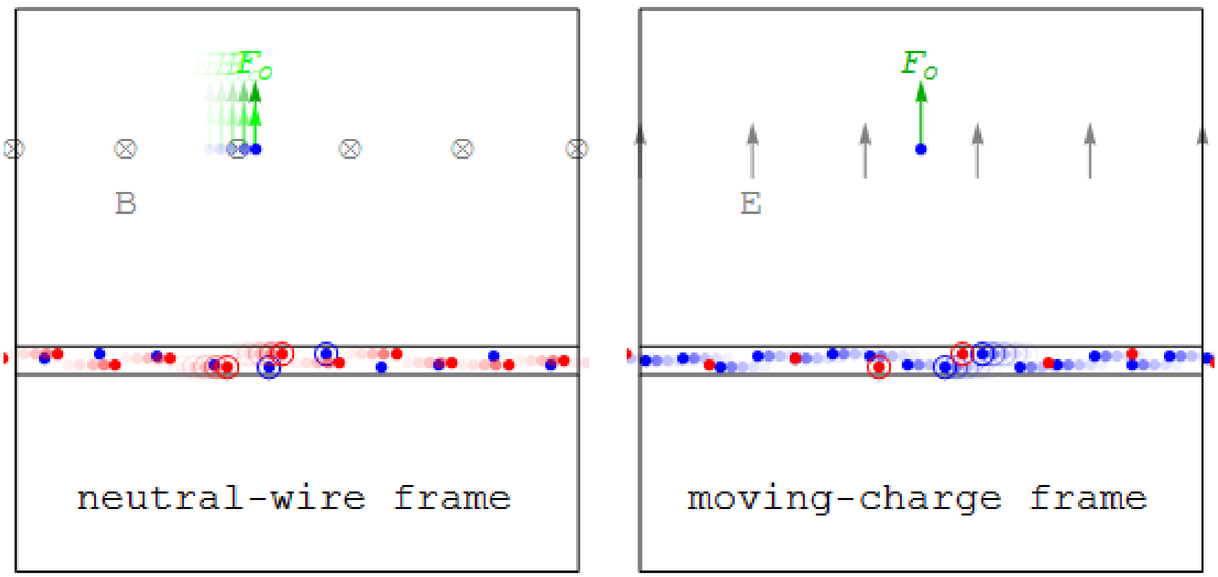

FIG. 3. Two views of proper force on a moving charge from a neutral current-carrying wire, with 40 millisecond time-steps between after-images. The shorter light-arrow in the wire-frame is the coordinate-force $f \equiv dp/dt = F_o/\gamma_\perp$. Effects of the depicted forces on the charge-motion are ignored, as is the B-field in the moving-charge frame which has no effect.

TABLE III. Relationships between variables for acceleration in (1+1)D flat-spacetime: Here τ is traveler-time elapsed from "turnaround" for as long as proper acceleration α doesn't change. The right arrow $\to$ shows simplification when $\alpha\tau \ll c$.

4-vector	invariants	time-components/c	space-components
acceleration	$\alpha \equiv \frac{\Sigma F_o}{m}$	$\frac{d\gamma}{d\tau} = \frac{\mathrm{P}}{mc^2} = \frac{\gamma P}{mc^2} = \frac{\alpha}{c}\sinh\left[\frac{\alpha\tau}{c}\right] \to \left(\frac{\alpha}{c}\right)^2\tau$	$\frac{dw}{d\tau} = \frac{\Sigma F}{m} = \frac{\gamma\Sigma f}{m} = \alpha\cosh\left[\frac{\alpha\tau}{c}\right] \to \alpha$
velocity	c	$\gamma \equiv \frac{dt}{d\tau} = \frac{E}{mc^2} = \sqrt{1+(\frac{w}{c})^2} = \cosh\left[\frac{\alpha\tau}{c}\right] \to 1$	$w \equiv \frac{dx}{d\tau} = \frac{p}{m} = \gamma v = c\sinh\left[\frac{\alpha\tau}{c}\right] \to \alpha\tau$
coordinate	τ	$t = \frac{c}{\alpha}\sinh\left[\frac{\alpha\tau}{c}\right] \to \tau$	$x = \frac{c^2}{\alpha}\left(\cosh\left[\frac{\alpha\tau}{c}\right] - 1\right) \to \frac{1}{2}\alpha\tau^2$

empirical observation exercise for students interested in the electrostatic origins of the magnetic force between moving charges. In essence, students are asked to take data in real time from animations (cf. Fig. 3) showing neutral-wire and moving-charge perspectives on the proper-force felt by the moving charge.

Simple ratios (in either space or time) allow students to quantify the length-contraction, the currents and charge densities from these two perspectives, and a variety of other physical quantities. In order to see significant differences in these quantities from the two perspectives, of course, charge velocities have to be relativistic. Since velocities are also perpendicular to observed forces, a significant difference between the coordinate-force observed in the neutral wire frame, and the proper-force felt by the moving charge, also shows up.

VI. DISCUSSION

As mentioned above, extended arrays of synchronized clocks are difficult to come by in curved spacetime (cf. the relativistic corrections needed to make global positioning estimates accurate). They are perhaps even more difficult to come by on accelerated platforms (cf. discussions of accelerated-frame "Rindler coordinates").

"Lorentz-transform first" analyses of any-speed motion of course require at least two relativistically co-moving frames of synchronized clocks. No wonder accelerated motion is of little interest in that context.

"Metric-first" approaches require only one such map-frame, since proper-time on traveler clocks is a frame-invariant. The integrals of constant proper-acceleration, especially in (1+1)D e.g. as $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x$ where $\eta \equiv \sinh[\frac{w}{c}]$, are also quite manageable. As shown Table III, which is a (1+1)D version of Tables I and II combined, the general *magnitude-inequality* between coordinate-force $\vec{f} \equiv d\vec{p}/dt$ (where we are using the relativistic momentum $\vec{p}$) and proper-acceleration $\vec{\alpha}$, namely $|\Sigma\vec{f}| \leq |m\vec{\alpha}|$, also becomes the more familiar-looking *signed-equality* $\Sigma f = m\alpha$.

The approach also works in curved-spacetime. Table IV illustrates for the "radial-only" Schwarzschild case using the exact Lorentz-factor from Hartle[4], even though the integration (even in the Newtonian case) is simplest if we can ignore variations of g with r. The competition between velocity-related, and gravitational, time-dilation e.g. for GPS-system orbits is nonetheless quite clear.

TABLE IV. Relationship between variables for acceleration in (1+1)D gravity: Here τ is traveler-time from "turnaround" for fixed proper acceleration, while as usual $g \equiv \frac{GM}{r^2}$ and $r_s \equiv \frac{2GM}{c^2}$. Here $\simeq$ neglects changes in g and $\rightarrow$ assumes that $\alpha\tau \ll c$.

4-vector	invariants	time-components/c	space-components
acceleration	$\alpha \equiv \frac{\Sigma F_o}{m}$	$\frac{d\gamma}{d\tau} = \frac{\mathrm{P}}{mc^2} = \frac{\gamma P}{mc^2} \simeq \frac{\alpha - g}{c} \sinh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow \left(\frac{\alpha-g}{c}\right)^2 \tau$	$\frac{dw}{d\tau} = \frac{\Sigma F}{m} = \frac{\gamma \Sigma f}{m} \simeq (\alpha - g)\cosh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow (\alpha - g)$
velocity	c	$\gamma \equiv \frac{dt}{d\tau} = \frac{E}{mc^2} = \gamma_r\sqrt{1 + \left(\frac{\gamma_r w}{c}\right)^2} \rightarrow \gamma_r \equiv \sqrt{\frac{1}{1-\frac{r_s}{r}}}$	$w \equiv \frac{dr}{d\tau} = \frac{p}{m} = \gamma v \simeq c \sinh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow (\alpha - g)\tau$
coordinate	τ	$t \simeq \frac{c}{\alpha - g}\sinh\left[\frac{(\alpha-g)\tau}{c}\right] \rightarrow \tau$	$r \simeq \frac{c^2}{\alpha-g}\left(\cosh\left[\frac{(\alpha-g)\tau}{c}\right] - 1\right) \rightarrow \frac{1}{2}(\alpha-g)\tau^2$

Just as in flat-spacetime, the metric equation in general associates a set of $\{t, x, y, z\}$ bookkeeper-coordinates with each event. In the Schwarzschild case, however, clocks can only be synchronized at fixed-r. Hence a radar-time model[11] (or some such) of extended-simultaneity might be needed to answer the question "What time is it now at radius r?"

The good news for the case of Schwarzschild (and other steady-state metrics) is that $\gamma \equiv \frac{dt}{d\tau} = \frac{E}{mc^2}$ can be defined regardless of one's model for extended-simultaneity. Although in general momentum $\vec{p} \equiv \frac{d\vec{x}}{d\tau}$ remains synchrony-free, definitions of synchrony-dependent energy may encounter significant complication when the bookkeeper time-derivative $\frac{dt}{d\tau}$ becomes dependent on extended-simultaneity.

We further show that frame-invariance (where all frames agree) is quite valuable for illustrations. The synchrony-free nature of proper-velocity and momentum, as well as of force-components described as derivatives using proper-time τ instead of map-time t, also lead to a simpler and more robust picture of accelerated motion when examined from the point of view of the accelerated traveler.

ACKNOWLEDGMENTS

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Appendix A: 3-vector relativity

By way of example, A. P. French[1] examines coordinate-acceleration components with respect to coordinate-velocity, J. D. Jackson's relativity chapters[14] do an excellent job at showing both 3-vector and Lorentz-covariant (4-vector) versions of the way that electro-magnetic proper forces work, and D. G. Messerschmitt[15] has examined scaling relations for proper-acceleration from a modern engineering perspective. Each of these has focused on components relative to the direction of motion, rather than relative to the direction of proper acceleration e.g. of a rocketship which has local control of its direction of thrust.

One must of course use caution in using relativistic 3-vectors (especially in curved-spacetime and accelerated-frames) because many implicit Newtonian assumptions are no longer valid. One may also encounter dissonance from uni-directional "simplifications", like the breakdown of relativistic-momentum $\vec{p}$ into a product of "relativistic-mass" γm and coordinate-velocity $\vec{v} \equiv d\vec{x}/dt$, rather than into a product of frame-invariant rest-mass m and synchrony-free proper-velocity $\vec{w} \equiv d\vec{x}/d\tau = \gamma\vec{v}$.

In flat spacetime, quantities demoted to the status of *frame-variant* or "effective for the indicated frame only" (eifo[13]) include:

- coordinate-time t in comparison to the frame-invariant proper-time τ elapsed e.g. on a traveling clock,

- simultaneity whose frame invariance may be (only temporarily) put aside by using the metric equation to select but one "book-keeper" definition of simultaneity, and

- force $\Sigma\vec{F} \equiv d\vec{p}/d\tau$ which we deal with by introducing the net felt or "proper-force" $\Sigma\vec{F}_o$ (from the traveler's perspective) whose magnitude (like proper-time τ and lightspeed c) is frame-invariant.

Note that rates of energy-change $dE/d\tau$ are frame-variant even at low speeds: For instance, $dE/d\tau$ is always zero in the rest-frame (as well as the "tangent free-float-frame") of an accelerated object even if energy is rapidly changing from the vantage point of other frames.

In curved-spacetimes and accelerated-frames, key concepts include the added ideas of:

- bookkeeper coordinates in the metric-equation which may be chosen for convenience but might not permit extended networks of synchronized clocks,

- vectors as locally-defined tangents[3] instead of as lines between points A and B,

- free-float (geodesic) trajectories and "tangent free-float-frames" in particular, and

- geometric (connection-coefficient) forces as distinct from proper-forces.

As discussed in the article, of course, the latter are also not new to students of low-speed physics if "inertial forces" like centrifugal and Coriolis have been studied.

These caveats in mind, familiar relations in scalar and/or 3-vector form can often be written with an additive $(\gamma-1)$ term for the eifo-correction. For instance, the motion-related eifo-correction to frame-invariant rest-energy mc^2 in flat spacetime is simply kinetic-energy $(\gamma-1)mc^2$, coordinate-time dilation for proper-time interval $\delta\tau$ may be written as $\delta t = \delta\tau + (\gamma-1)\delta\tau$, and vector length-contraction for proper-length interval $\vec{L}_o$ is

$$\vec{L} = \vec{L}_o + \frac{(\gamma-1)}{\gamma}\vec{L}_{o_{||\vec{w}}}, \tag{A1}$$

where the subscript $||\vec{w}$ selects only that component of $\vec{L}_o$ which is parallel to proper-velocity w.

More generally, for 4-vector $\{X_t, \vec{X}\}$ the Lorentz boost to a primed-frame moving at proper-velocity $\vec{w}$ has time-component $X'_t = X_t - \frac{\vec{w}}{c}\cdot\vec{X} + (\gamma_w - 1)X_t$. The space-component is $\vec{X}' = \vec{X} - \frac{\vec{w}}{c}X_t + (\gamma_w - 1)\vec{X}_{||\vec{w}}$.

Since proper-acceleration is purely spacelike in the frame of the accelerated traveler, we can say that the time-component yields the flat spacetime work-energy expression $dE/d\tau = m\vec{\alpha}\cdot\vec{w} = \Sigma\vec{F}_o\cdot\vec{w}$. The space-component says that the frame-variant force can be expressed in terms of proper-acceleration[16] and proper-velocity as:

$$\Sigma\vec{F} = \frac{d\vec{p}}{d\tau} = m\frac{d\vec{w}}{d\tau} = m\vec{\alpha} + (\gamma-1)m\vec{\alpha}_{||\vec{w}}, \tag{A2}$$

The second term here is a correction to the net proper-force $\Sigma\vec{F}_o = m\alpha$ that (like the only term in the work-energy expression since "proper-power" is always zero) allows one to determine the net-force $\Sigma\vec{F} \equiv d\vec{p}/d\tau$ from the map-frame perspective.

Note that since proper-velocity uses a time-variable localized to the traveler (and hence does not require synchronized map-clocks along the traveler's trajectory), these 3-vector expressions may be useful *locally* in curved as well as flat spacetime settings, provided that we have a definition for $\gamma \equiv dt/d\tau$ (from the metric) and hence an effective value for traveler total energy $E = \gamma mc^2$.

In flat spacetime, where the metric tells us that $\gamma \equiv dt/d\tau = \sqrt{1+(w/c)^2}$, one can obtain the energy-integral differential equation:

$$\frac{c^2}{\alpha}\ddot{\gamma} = \left(\frac{1+\gamma+(\frac{c}{\alpha}\dot{\gamma})^2}{1+\gamma}\right)\alpha \tag{A3}$$

where the dot refers to differentiation with respect to proper-time τ, and $w_{||} = (c^2/\alpha)d\gamma/d\tau$. This integrates pretty quickly to the contents of Table II. Table III entries then follow directly for the (1+1)D case by letting $\gamma_o \to 1$.

For the Schwarzschild potential, $dt/d\tau$ becomes $\gamma_r\sqrt{(1+(\gamma_r w/c)^2)}$ if $\gamma_r \equiv 1/(1-r_s/r)$ and r_s is the event-horizon radius $2GM/c^2$. Application of equation A2 then *only qualitatively* yields the approximate relationships in Table IV.

Note also that the rescaled velocity-term in equation A2 is reminescent of the factor $(\gamma_{BC} + (\gamma_{AB} - 1)\vec{w}_{\mathrm{BC}_{||\vec{w}_{\mathrm{AB}}}})$ that rescales (in magnitude only) the out-of-frame proper-velocity $\vec{w}_{AB} \to (\vec{w}_{AB})_C$ when calculating relative proper-velocity 3-vectors by 3-vector addition:

$$\vec{w}_{AC} \equiv \gamma_{AC}\vec{v}_{AC} = (\vec{w}_{AB})_C + \vec{w}_{BC}. \tag{A4}$$

Thus a focus on frame-invariant and synchrony-free variables might help *cautiously* open the door to a wider range of "dimensioned 3-vector" relativistic explorations.

Finally, let's examine the connection of these equations to the Lorentz-equation for electromagnetic proper-force, which underpins Fig. 3 as well as most proper-forces that we encounter in everyday life. It is also a prototype for the Maxwell-like equations that underpin field-mediated proper-forces in general. In terms of electric $\vec{E}$ and magnetic $\vec{B}$ fields in a frame with respect to which a charge Q is moving at proper-velocity $\vec{w} = \gamma\vec{v}$, we can use the Lorentz-transform equations (SI-units version[14]) for the electric field in the "primed" frame of charge Q to write the proper-force as:

$$\Sigma\vec{F}_o = Q\vec{E}' = Q\vec{E}_{||\vec{w}} + \gamma Q(\vec{E}_{\perp\vec{w}} + \vec{v}\times\vec{B}) \tag{A5}$$

Although the force on our moving charge (as a rate of momentum change) is in general frame-variant, all observers (traveling at any speed even in curved spacetime) should be able to agree on the proper-force and proper-acceleration that the charge is experiencing. Putting this general expression, for the net frame-invariant proper-force on a moving charge, into the expression for net frame-variant force above gives us:

$$\Sigma\vec{F} \equiv \frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt} = \gamma Q(\vec{E} + \vec{v} \times \vec{B}) \qquad \text{(A6)}$$

This Lorentz-force expression, here obtained from the electrostatic definition of $\vec{E}$ and the field transformation-rules, illustrates how a "magnetic field" that yields a force perpendicular to velocity may serve as a natural complement to any static "proper-force field".