Parameterizing the proper-acceleration three-vector

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Abstract

The implicit idea of global time adds dissonance to the bridge between Newtonian dynamics and high-speeds/curved-spacetimes. However Newtonian use of 3-vectors connects naturally to a "traveler-point dynamics" that starts with "minimally frame-variant" proper-time/velocity/acceleration. This allows one to express the introphysics equations of constant acceleration as a limiting case of *characteristic-time* equations which work well at both high speeds and in curved spacetimes, provided that we (as usual) approximate geometric forces (like gravity and centrifugal which are invisible to cell-phone accelerometers) as proper forces. This approach can help make engineering pedagogy "spacetime smart", and be useful in relativistic physics engines used for cinema, video games, and interstellar travel simulations.

CONTENTS

I. INTRODUCTION

Although hyperbolic motion was discussed by Hermann Minkowski[1] before he passed away in 1909 (and shortly thereafter by Max Born[2]), proper acceleration itself (as a 4-vector with but 3-vector components in the traveler frame) only comes up later albeit in a wide range of technical papers[3–6] and books[7]. Neither coordinate nor proper acceleration are often discussed in context of "Lorentz-transform based" special relativity (except perhaps to comment on the former's uselessness[8]), because accelerating an extended Lorentz frame automatically de-synchronizes its clocks[9].

However, if bookkeeper coordinates of the metric are used to define simultaneity as well as the distance between separated events, the local traveler-point parameterization makes proper acceleration (as a 3-vector[10] with its frame-invariant magnitude) accessible without need for the full 4-vector formalism of curved spacetimes (which looks past the symmetry break between space and time experienced by local observers). Subsequent to *Spacetime Physics*[11], constant proper acceleration round trips and related formulae (mostly with one spatial dimension) are discussed in various papers for the physics education community[12–15].

There are many ways to describe accelerated motion in spacetime, because one can choose between using:

- time on map-clocks or traveler-clocks;
- velocity as *distances* on map[8,16] or traveler[8] yardsticks *per unit time* on map[8] or traveler[16] clocks;
- acceleration as *second derivatives of distance* on map[8] or traveler yardsticks *with respect to time* on map[8] or traveler[15,17] clocks.

In (3+1)D one can also resolve components with respect to the directon of either: (i) the "heading" or velocity[8,17], (ii) the "traveler's map-acceleration"[17] or frame-variant force, (iii) the coordinate-acceleration[18], or (iv) the proper-acceleration[15].

Only one of these descriptions works robustly (cf. Fig. 1) for 3-vectors in *curved* spacetime, namely that using "traveler-point" variables, namely frame-invariant *proper time* τ, synchrony-free[16] *proper velocity*[19] $\vec{w} \equiv \delta\vec{x}/\delta\tau$, and magnitude-invariant *proper acceleration* $\vec{\alpha}$. This strategy mirrors the "one-map two-clock" form of the metric-equation itself, whose zeroeth through second derivative versions (with respect to traveler-time) define frame invariants in terms of displacements on bookkeeper or map yardsticks and times elapsed on map clocks. Work with components referenced to the direction of the constant proper-acceleration looks particularly promising.

We describe a parameterization of these flat-space equations of motion in terms of a *characteristic* "traveler-time from turnaround"

$$\tau_o \equiv \frac{c}{\alpha}\sqrt{\tfrac{\gamma_o - 1}{2}}, \tag{1}$$

after which the exponential or "e-folding" effects of constant proper-acceleration's hyperbolic trig-functions start to "kick in". Here c is lightspeed, α is the proper-acceleration magni-

TABLE I. Constant proper acceleration $\vec{\alpha}$ in flat spacetime: Here τ is traveler-time from "turnaround" when proper velocity $\vec{w} = \vec{w}_o \perp \vec{\alpha}$, initial aging-factor $\gamma_o = \sqrt{1+(w_o/c)^2}$), and characteristic time is $\tau_o \equiv \sqrt{\frac{1}{2}(\gamma_o+1)}c/\alpha$.

4-vector V	invariant	scalar timelike component V^t/c	3-vector spacelike component $\vec{V}$
coordinate $\{X^t, \vec{X}\}$	proper time τ $\sqrt{g_{\mu\nu}X^\mu X^\nu}$	$t = \frac{1}{2}\frac{(w_o/c)^2}{\gamma_o+1}\tau + \left(\frac{\alpha\tau_o}{c}\right)^2 \tau_o \sinh\left[\frac{\tau}{\tau_o}\right]$ $\vec{w}_o = 0 \rightarrow t = \tau_o \sinh\left[\frac{\tau}{\tau_o}\right]$ $w \ll c \rightarrow t \simeq \tau$	$\vec{r} =$ $\frac{1}{2}(\tau + \tau_o \sinh\left[\frac{\tau}{\tau_o}\right])\vec{w}_o + \tau_o^2(\cosh\left[\frac{\tau}{\tau_o}\right] - 1)\vec{\alpha}$ $\vec{w}_o = 0 \rightarrow \vec{r} = \tau_o^2\left(\cosh\left[\frac{\tau}{\tau_o}\right] - 1\right)\vec{\alpha}$ $w \ll c \rightarrow \vec{r} \simeq \vec{w}_o\tau + \frac{1}{2}\vec{\alpha}\tau^2$
velocity $\{U^t, \vec{U}\}$	lightspeed c $\sqrt{g_{\mu\nu}U^\mu U^\nu}$	$\frac{dt}{d\tau} \equiv \gamma = \frac{1}{2}\frac{(w_o/c)^2}{\gamma_o+1} + \left(\frac{\alpha\tau_o}{c}\right)^2 \cosh\left[\frac{\tau}{\tau_o}\right]$ $\vec{w}_o = 0 \rightarrow \gamma = \cosh\left[\frac{\tau}{\tau_o}\right]$ $w \ll c \rightarrow \gamma \simeq 1 + \frac{1}{2}\left(\frac{w}{c}\right)^2$	$\frac{d\vec{r}}{d\tau} \equiv \vec{w} =$ $\frac{1}{2}(1 + \cosh\left[\frac{\tau}{\tau_o}\right])\vec{w}_o + \tau_o \sinh\left[\frac{\tau}{\tau_o}\right]\vec{\alpha}$ $\vec{w}_o = 0 \rightarrow \vec{w} = \tau_o \sinh\left[\frac{\tau}{\tau_o}\right]\vec{\alpha}$ $w \ll c \rightarrow \vec{w} \simeq \vec{w}_o + \vec{\alpha}\tau$
acceleration $\{A^t, \vec{A}\}$	proper acceleration α $\sqrt{-g_{\mu\nu}A^\mu A^\nu}$	$\frac{\delta^2 t}{\delta\tau^2} = \frac{\delta\gamma}{\delta\tau} = (\frac{\alpha}{c})^2\tau_o \sinh\left[\frac{\tau}{\tau_o}\right]$ $\vec{w}_o = 0 \rightarrow \frac{\delta\gamma}{\delta\tau} = \frac{1}{\tau_o}\sinh\left[\frac{\tau}{\tau_o}\right]$ $w \ll c \rightarrow \frac{\delta\gamma}{\delta\tau} \simeq (\frac{\alpha}{c})^2\tau$	$\frac{\delta^2\vec{x}}{\delta\tau^2} = \frac{\delta\vec{w}}{\delta\tau} =$ $\frac{1}{2}(\sinh\left[\frac{\tau}{\tau_o}\right]/\tau_o)\vec{w}_o + \cosh\left[\frac{\tau}{\tau_o}\right]\vec{\alpha}$ $\vec{w}_o = 0 \rightarrow \frac{\delta\vec{w}}{\delta\tau} = \cosh\left[\frac{\tau}{\tau_o}\right]\vec{\alpha}$ $w \ll c \rightarrow \frac{\delta\vec{w}}{\delta\tau} \simeq \vec{\alpha}$

TABLE II. Traveler-point dynamics in flat spacetime: Conserved quantities energy $E = \gamma mc^2$ and momentum $\vec{p} = m\vec{w}$, where differential-aging factor $\gamma \equiv \delta t/\delta\tau$, proper velocity $\vec{w} \equiv \delta\vec{r}/\delta\tau \equiv \gamma\vec{v}$, coordinate acceleration $\vec{a} \equiv \delta\vec{v}/\delta t$.

relation	(1+1)D	(3+1)D	$w \ll c \rightarrow \gamma \simeq 1$
momentum $\vec{p}$	$\vec{p} = m\vec{w} = m\gamma\vec{v}$	$\vec{p} = m\vec{w} = m\gamma\vec{v}$	$\vec{p} \simeq m\vec{v}$
energy E	$E = \gamma mc^2$	$E = \gamma mc^2$	$E \simeq mc^2 + \frac{1}{2}mv^2$
felt ($\vec{\xi}$) $\leftrightarrow$ map-based ($\vec{f}$) force conversions	$\vec{f} = \vec{\xi}$	$\vec{\xi} = \vec{f}_{\vert\vert\vec{w}} + \gamma\vec{f}_{\perp\vec{w}}$ $\vec{f} = \vec{\xi}_{\vert\vert\vec{w}} + \vec{\xi}_{\perp\vec{w}}/\gamma$	$\vec{f} \simeq \vec{\xi}$
map-based force:momentum felt force:acceleration	$\Sigma\vec{f} = \delta\vec{p}/\delta t = m\vec{\alpha}$	$\Sigma\vec{f} = \delta\vec{p}/\delta t$ $\Sigma\vec{\xi} = m\vec{\alpha}$	$\Sigma\vec{f} = \delta\vec{p}/\delta t \simeq m\vec{a}$
action-reaction	$\vec{f}_{\rm AB} = -\vec{f}_{\rm BA}$	$\vec{f}_{\rm AB} = -\vec{f}_{\rm BA}$	$\vec{f}_{\rm AB} = -\vec{f}_{\rm BA}$
work-energy	$\delta E = \Sigma\vec{f} \cdot \delta\vec{x}$	$\delta E = \Sigma\vec{\xi} \cdot \delta\vec{x}$	$\delta E \simeq \Sigma\vec{f} \cdot \delta\vec{x}$

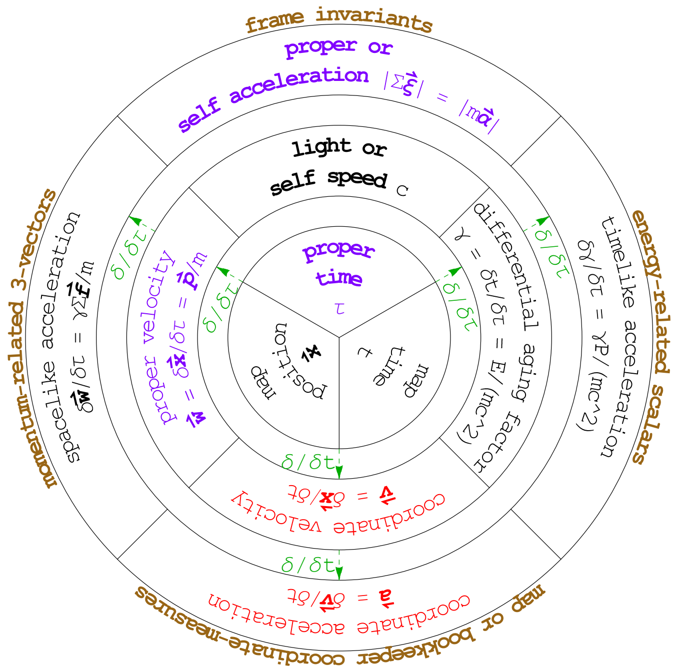

FIG. 1. Scalar and 3-vector parameters for characterizing motion using a single metric: Practical use of the red parameters may require synchronized clocks, e.g. to measure map time at different positions, while the indigo "traveler-point" parameter set is synchrony-free, and has minimal frame-variance.

tude, and γ_o is the differential-aging (Lorentz) factor "at turnaround", when the velocity-component parallel to that acceleration reverses sign. The result is a set of scalar (time, energy, power) and 3-vector (position, momentum, force) equations that reduce seamlessly to the familiar low-speed equations. Their application, initially in flat-spacetime simulations, is also discussed because (in addition to the hyperbolic trig functions) there are some important differences when operating in this metric-first[20] or one-map two-clock[21] world.

II. METRIC INVARIANTS AND THEIR SOLUTIONS

We start with the flat-space interval or metric equation in Cartesian form, as a spacetime Pythagoras' theorem with "time-like" hypoteneuse:

$$(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2 - (\delta y)^2 - (\delta z)^2 \tag{2}$$

where as usual t represents scalar map-time and 3-vector $\vec{x}$ represents map-position, while frame-invariant proper time (elapsed on the clocks of our accelerated object) is represented by

τ, and c is the space/time conversion factor e.g. the number of meters (namely 299,792,458) defined to be contained in a second.

Applying the metric equation to the first proper time derivative of each displacement component gives a velocity-component relation:

$$c^2 = (c\gamma)^2 - (w_x)^2 - (w_y)^2 - (w_z)^2 \tag{3}$$

where differential-aging (Lorentz) factor $\gamma \equiv \delta t/\delta\tau$ and proper velocity $\vec{w} \equiv \delta\vec{x}/\delta\tau$. Here of course the "synchrony-free" three-vector proper-velocity $\vec{w}$ is also the traveling object's 3-vector momentum $\vec{p}$ per unit mass, while the scalar differential-aging factor $\gamma \geq 1$ is the traveling object's total energy $E = mc^2 + K$ divided by mc^2. In curved spacetimes and accelerated frames, in addition to rest and kinetic energy this Lorentz factor γ will also include terms for potential energies associated with improper "geometric" (i.e. connection-coefficient) forces like gravity and centrifugal.

This differential equation yields the familiar flat-space relationships for Lorentz-factor, namely $\gamma = \sqrt{1+(w/c)^2} = 1/\sqrt{1-(v/c)^2}$ where v is the magnitude of coordinate velocity $\vec{v} \equiv \delta\vec{x}/\delta t$, and shows that c (now seen to be the magnitude of <u>all</u> 4-vector velocities) is a true frame-invariant just as are proper-time intervals like $\delta\tau$. In other words, all frames of reference agree on their values.

Applying the metric equation to the second proper-time derivative of each displacement component gives an acceleration component relation that can be used to define a third frame-invariant, namely the proper acceleration 3-vector's *magnitude* α, defined by:

$$-\alpha^2 = (c\tfrac{\delta\gamma}{\delta\tau})^2 - (\tfrac{\delta w_x}{\delta\tau})^2 - (\tfrac{\delta w_y}{\delta\tau})^2 - (\tfrac{\delta w_z}{\delta\tau})^2. \tag{4}$$

In terms of conserved quantities, we'll refer to spatial 3-vector $\frac{\delta\vec{w}}{\delta\tau}$ as the "net frame-variant force" $\Sigma\vec{F} \equiv \delta\vec{p}/\delta\tau$ *per unit mass*, and the temporal scalar $\frac{\delta\gamma}{\delta\tau}$ as the "fractional empowerment" or rate of energy change $P \equiv \delta E/\delta\tau$ (e.g. in watts) divided by mc^2.

Note that the proper acceleration 4-vector becomes a pure 3-vector $\vec{\alpha}$ in the frame of the traveling object (and in free-float frames "tangent" to the traveler's world line), since γ has "bottomed out" at 1 so that the time component $c\frac{\delta\gamma}{\delta\tau}$ has vanished. Because the foregoing equations must be true instantaneously (with simultaneity defined by our choice of a "free-float" or inertial origin for map-time t and spatial 3-vector $\vec{x}$), we can think of equations 3 and 4 as a set of differential equations with which we can track an object's unfolding motion if both the magnitude and direction of that proper-acceleration 3-vector stays unchanged.

With help from various parameterizations of the (3+1)D acceleration equations in the literature, a second-order ordinary differential equation $\ddot{\gamma} - \dot{\gamma}^2/(1+\gamma) = (\alpha/c)^2$ in the traveling object's Lorentz factor, with a closed form solution, was obtained. The characteristic time parameterization of that solution illustrated in Table I emerged later. This allows for an elegant scalar/3-vector representation, and can be substituted (numerically with ease, as shown in Table IV) into the differential equations above to show that it works.

Note that we are here using the subscript "o" in two different ways. One is to flag quantities, like γ_o and $\vec{w}_o$, associated with a traveler's motion "at turnaround" when τ and t equal zero in our constant acceleration model. The other is to flag special quantities, and in particular the "characteristic time" τ_o pertaining to accelerated motion *at any time* from the vantage point of a specific bookkeeper frame.

III. WORK-ENERGY AND NEWTON'S LAWS

This parameterization generalizes the low-speed 3-vector equations for use at high speeds in flat spacetime, but it also specializes general relativity's 4-vector equations of motion with help from "synchrony free" parameters which are as frame-invariant as possible. This is easiest to see with the generalization of the work-energy theorem which shows up in the observation from Table I that $\delta E/\delta\tau = m\vec{\alpha}\cdot\vec{w}$. This is special case of the more general work-energy relation, which says that $\delta E = \delta\gamma mc^2 = \Sigma\vec{\xi}\cdot\delta\vec{r}$, where $\Sigma\vec{\xi}$ is the net frame-invariant "felt force" $\vec{\xi}$ experienced by our traveler such that $\Sigma\vec{\xi} = m\vec{\alpha}$. These $\vec{\xi}$ are specifically those "proper forces" whose sum is detectable by accelerometers fixed to our traveler.

In flat spacetime, of course $E = \gamma mc^2$ includes only rest energy mc^2 and kinetic energy $(\gamma - 1)mc^2$, but in curved spacetimes it also includes potential energies associated with local connection-coefficient or "geometric" forces like $-GMm/r$ for gravity around a planet, $m\alpha\Delta x$ for a long vehicle accelerating in the x-direction, and $-\frac{1}{2}m\omega^2r^2$ for the "artificial gravity" in a centrifuge. Thus we have a work-energy theorem that not only works at any speed, but that connects the potential energies associated with geometric forces to the differential-aging factor γ on the non-force side of the equation. By moving these potentials to the proper-force side of the equation, we get the local approximations commonly used in curved spacetime and accelerated frames here on earth. The right-hand side of this work-energy theorem in flat space-time also allows one to experimentally track the energy-related quantities on the right-hand side of Fig. 1, without the need for synchronized clocks.

We now turn to the momentum-related 3-vectors on the left-hand side of Fig. 1. Rates of energy-change are frame-variant at any speed, but at high speed rates of momentum-change (even in flat spacetime) become frame-variant as well. Nonetheless, proper acceleration retains an intimate connection to these rates of change.

To see this we define a version of the "map-based force" $\vec{f}$ acting on a traveling object with fixed rest-mass m in terms of map-time, namely $\Sigma\vec{f} \equiv m\delta\vec{w}/\delta t = \delta\vec{p}/\delta t$. Momentum conservation from a given map-frame's perspective also includes action-reaction i.e. $\vec{f}_{AB} = -\vec{f}_{BA}$. Thus Newton's law takes the same form at any speed in flat spacetime as it does in the Newtonian low-speed approximation. We summarize these relationships in Table II using a focus on the simple distinction between frame-invariant "felt force" (denoted by $\vec{\xi}$), connected directly to proper acceleration, and "map-based force" $\vec{f}$ connected directly to rates of momentum change in the map-frame.

For instance, in (1+1)D spacetime we can write Newton's 2nd law as simply $\Sigma\vec{f} = m\vec{\alpha}$, provided that we carefully use forces $\vec{f}_i$ here defined from the vantage-point of our bookkeeper metric or map frame. In (3+1)D the 2nd law becomes more complicated. This is more clearly seen using scaling relations for components parallel and perpendicular to the instantaneous proper velocity[17], which allow us to say that $\vec{f} = \vec{\xi}_{||\vec{w}} + (1/\gamma)\vec{\xi}_{\perp\vec{w}}$. In general, therefore, the acceleration form of Newton's 2nd law is only a magnitude inequality, namely $0 \leq |\Sigma\vec{f}| \leq |m\vec{\alpha}| = |\vec{\xi}|$, where as before the right-hand side of this inequality is seen to have the same value from all (even accelerated frame) points of view.

We should also mention that one consequence of the inequality between proper (or felt) force $\vec{\xi}$ and the corresponding map-based force $\vec{f}$ is that the latter naturally breaks down into a sum of static and kinetic components, like $\vec{f}_E \equiv \vec{\xi}_{||\vec{w}} + \gamma\vec{\xi}_{\perp\vec{w}} = \gamma^2 m\vec{a}$ and $\vec{f}_B \equiv -\gamma(v/c)^2\vec{\xi}_{\perp\vec{w}}$ when to the traveling object there are no hidden kinetic components (like B-fields unfelt in the object frame). This breakdown is especially useful when proper forces may be exerted by oppositely-signed force carriers. The distinction is so useful, in fact, that it was discovered (along with an elegant notation for dealing with it) in the mid-19th century while modeling electrostatic and magnetic (i.e. kinetic-component) forces, well before the intimate connection between space and time was uncovered.

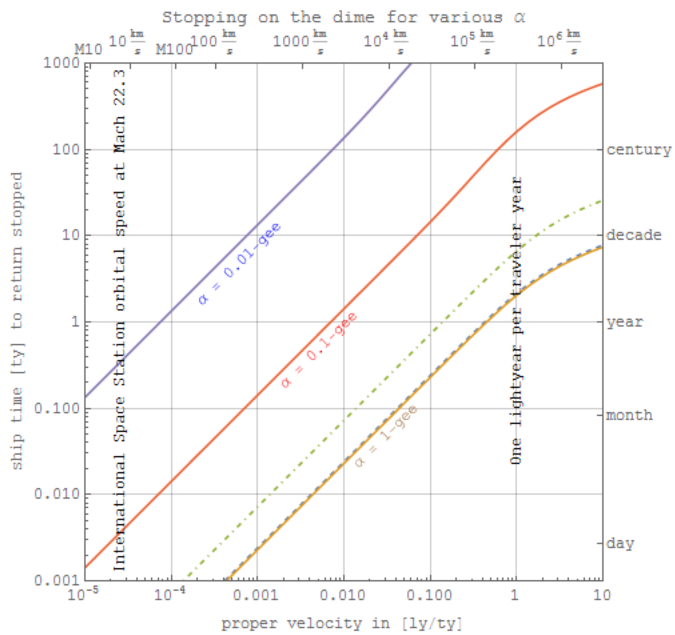

FIG. 2. It takes time to stop on a dime, not to mention fuel. This is true at any speed, but especially at proper-speeds greater than 1[ly/ty], where it takes more than 2 traveler years of 1-gee thrust to return "stopped" at your current location.

IV. USES IN SIMULATION AND IN PEDAGOGY

Although the equations in Table I and II look remarkably like the familiar ones that work only at low speed, there are important differences in doing 3-vector dynamics with traveler-point variables at high speeds, and especially in curved spacetime or accelerated frames. Some arise from the structure of the equations themselves. For instance, determining the characteristic-time τ_o as well as the "turnaround parameters" γ_o and $\vec{w}_o$ given a nonzero initial velocity $\vec{w}_1$ not perpendicular to $\vec{\alpha}$, as well as the "proper-time offset" or initial time-from-turnaround (call it τ_1) is more complicated than we are used to when working with Newtonian approximations at low speed.

Fortunately these have closed form solutions. The always-positive characteristic time τ_o (this is not a time on any clock) can be obtained from $\vec{w}_1$ (with magnitude w_1 and angle θ with respect to $\vec{\alpha}$) using:

$$\tau_o = \frac{c}{\alpha}\frac{w_1}{\sqrt{w_1^2\cos[\theta]^2 + 2c^2(\gamma_1 - 1)\sin[\theta]^2}} \tag{5}$$

Here the current differential-aging (Lorentz) factor is as usual $\gamma_1 = \sqrt{1 + (w_1/c)^2}$. The

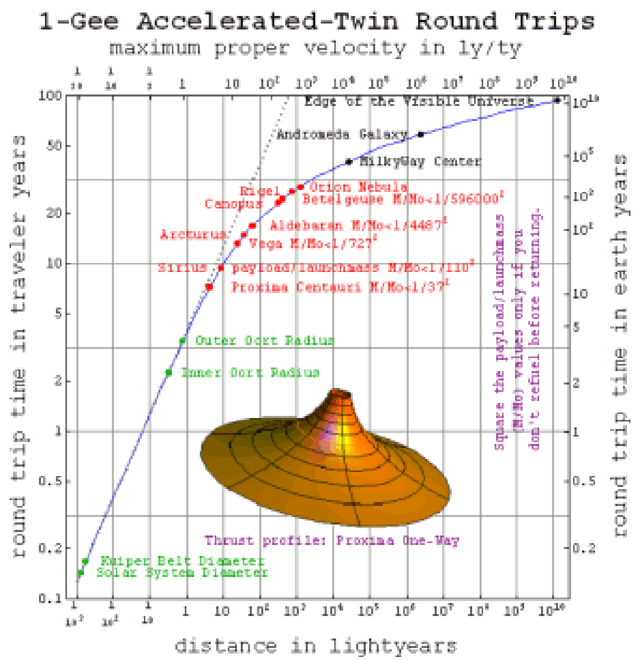

FIG. 3. For a given traveler-time relativity drastically increases the range for a 1-gee round-trip over the Newtonian model (dotted), but payload to launchmass ratios also drop off exponentially.

result is greater than or equal to c/α.

Given this, the time elapsed "since or until" turnaround becomes:

$$\tau_1 = \pm\tau_o \cosh^{-1}\left[(1+\gamma_1)\left(\frac{c}{\alpha\tau_o}\right)^2 - 1\right] \tag{6}$$

This is positive (i.e. since turnaround) when θ is acute (e.g. between $\pm\pi/2$), and negative (i.e. until turnaround) when θ is obtuse.

We of course also need an expression for the proper-velocity vector $\vec{w}_o$ (and perhaps Lorentz factor $\gamma_o = \sqrt{1+(w_o/c)^2)}$ at turnaound. This becomes:

$$\vec{w}_o = \left(2\alpha\tau_o\sqrt{\left(\frac{\alpha\tau_o}{c}\right)^2 - 1}\right)\hat{u}_1 \tag{7}$$

where $\hat{u}_1$ is a unit vector directed along the component of $\vec{w}_1$ which is perpendicular to $\vec{\alpha}$, e.g. equal to $\vec{w}_1 - w_1 cos[\theta]\vec{\alpha}/\alpha$ divided by its own magnitude.

Given these quantities, one can simply infer change in map-position at any traveler time $\Delta\tau$ elapsed from an arbitrary initial velocity $\vec{w}_1$ under constant proper acceleration $\vec{\alpha}$ by following:

$$\Delta\vec{x}[\Delta\tau] = \frac{\tau_o}{2}(\Delta\sinh[\frac{\tau}{\tau_o}])\vec{w}_o + \tau_o^2(\Delta\cosh[\frac{\tau}{\tau_o}])\vec{\alpha} \tag{8}$$

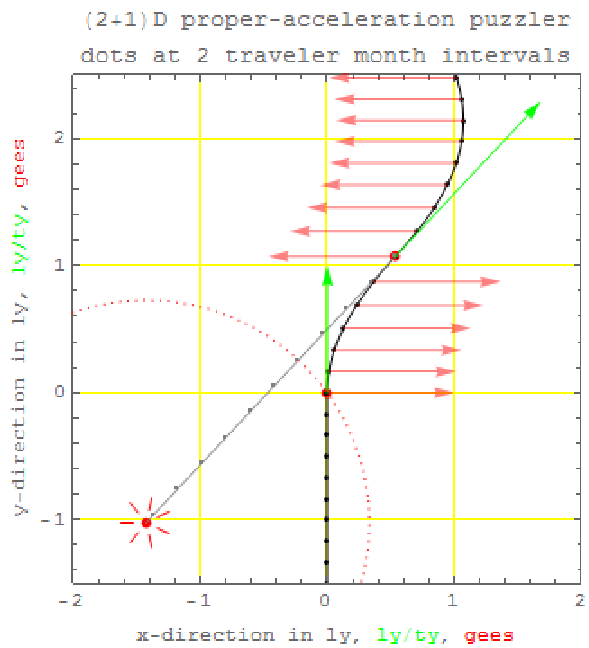

FIG. 4. Schematic of a starfleet battlecruiser's rendezvous with an enemy ship, discussed in the on-line supplment.

where the Δ in front of a hyperbolic trig function means the difference between the value at $\tau_1 + \Delta\tau$ and at τ_1. The traveler's new proper-velocity at the end of that interval is:

$$\vec{w}[\Delta\tau] = \frac{1 + \cosh[\frac{\tau_1+\Delta\tau}{\tau_o}]}{2}\vec{w}_o + \tau_o \sinh[\frac{\tau_1 + \Delta\tau}{\tau_o}]\vec{\alpha} \tag{9}$$

These quantities can then be used to extrapolate the constant proper-acceleration trajectory as far as we like (forward or backward) by $\Delta\tau$ from the current time τ_1 with respect to turnaround. For changing accelerations, one can also use this numerically to calculate trajectories as Riemann sums of small fixed-acceleration proper-time intervals.

This also suggests a more general problem in use of traveler-point variables, namely that they are specific to one traveler only. Moreover, at high speeds the complicated frame-dependence for forces, velocities, and times must be taken into account when changing bookkeeper frames, even if it is a free-float-frame in flat spacetime moving at a constant speed with respect to the original bookkeeper frame. Thus asking about the dynamics of one accelerated traveler from the perspective of another, for example, is a much bigger problem in (3+1)D than it would be with the Galilean low-speed approximations. The approach only avoids the problem of "whose simultaneity" and "whose yardsticks" by sticking with one bookkeeper metric only, and describing the motion using variables which are minimally frame-variant and synchrony free.

V. CONCLUSIONS

Physics engines for interstellar flight simulations e.g. in *Celestia* and *Orbiter*, for sci-fi movies or games, and even for research simulations may benefit from an approach which uses dimensioned variables and a familiar 3-vector context. There is also an appetite in the engineering community[17] for a more "spacetime smart" education, and of course students who are enthused about the challenges of space travel will be happy to know that the familiar low speed equations can be extended in a fairly natural way to high speeds, and numerically to curved spacetimes.

The parameterization makes accessible to intro-physics students problems like those illustrated in Figs. 2, 3, and 4. The last of these is discssued in more detail, with solutions, in the on-line supplement. Note that 1-gee acceleration is approximately 1[ly/y^2], so that the minimum (and unidirectional) value for characteristic time τ_o is $c/\alpha \simeq 1$ traveler year or $[ty]$. This both constrains "relativistically interesting" travel for humans to voyages measured in years, and really simplifies "back of envelope" calculations involving 1-gee acceleration.

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Appendix A: Supplementary Material

1. Intro to motion studies

In this section of the supplementary material we show introductory physics teachers and students how the minimally frame-variant "traveler-point variables" (*proper* time, velocity, and acceleration), used in the main paper to parameterize constant proper-acceleration, can also help put Newtonian motion studies into a relativity-smart but unit-rich 3-vector form.

a. start with a map

To describe motion we often first define a bookkeeper coordinate-system, or "map with metric", made up of distributed clocks as well as position markers that will be used to locally describe the map position $\vec{r} = \{x, y, z\}$ and map time t of events. This coordinate-system might be simply be a set of identical synchronized clocks separated by standardized yardsticks, although this is not always possible even here on earth e.g. since time's rate of passage increases with altitude i.e. when standing, your head ages faster than your feet.

When two separately-located events show the same time t on their bookkeeper clocks, we say that those events are simultaneous. Folks using a different map-frame or metric, even one simply moving at a fixed speed with respect to the first, may determine that those two events are not simultaneous i.e. that one event occurs before the other. Hence map-time, map-position, and the meaning of simultaneity are all defined by our choice of coordinate-system or metric.

b. traveler-point kinematics

Motion may be described most generally, e.g. at any speed and in curved spacetimes, by describing it from one traveling object's perspective. That is because every one (regardless of metric, including our traveler) measures the same value for the separation between events, or proper-time τ elapsed, on a traveling object's "world line" through spacetime. To work with many traveling objects at once, particularly at low speeds, it may be useful to pretend that elapsed map-time δt and all traveler-times elapsed ($\delta\tau$) are one and the same, but we don't want to do that just yet.

The increment of traveler-time $\delta\tau$ elapsed along a world line follows a kind of space-time Pythagoras' theorem. In its simplest form, this can be written as $(c\delta\tau)^2 = (c\delta t)^2 - \delta\vec{s}\cdot\delta\vec{s}$, where the 3-vector displacement-increment or hypotenuse $\delta\vec{s}$, as usual in Cartesian $\{x, y, z\}$ coordinates, obeys $\delta\vec{s}\cdot\delta\vec{s} = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$.

Here the spacetime constant c, often referred to as lightspeed or "the number of meters in a second" is, like traveler-time, also a frame-invariant i.e. something on which observers from all vantage points can agree. This follows because the metric equation also defines invariants for derivatives of the map-coordinates $\vec{r}$ and t, with respect to proper-time τ.

This allows us to define the "first-derivative" traveler-point velocity variables named proper-velocity $\vec{w} \equiv \delta\vec{r}/\delta\tau$ and differential aging-factor $\gamma \equiv \delta t/\delta\tau$, as well as "second-derivative" traveler-point velocity variables which, when combined, allow one to track the traveler's proper-acceleration 3-vector $\vec{\alpha}$, whose magnitude (like that of $\delta\tau$ and c) is also frame-invariant i.e. something on which observers in all frames reference can agree. That proper acceleration 3-vector is also the acceleration measured by the accelerometer in the traveler's smartphone.

Proper-velocity $\vec{w}$ does not require synchronized map-clocks for its measurement, and it is equal to the momentum $\vec{p}$ carried by our traveler per unit traveler rest-mass m. Note that any traveling object's rest-mass is also, in this context, a frame-invariant and something on which observers from all frames can agree. Unlike the "coordinate-velocity" $\vec{v} \equiv \delta\vec{r}/\delta t$ that we'll be using later in this course, proper-velocity also has no upper limit. This is great news for relativistic travelers, if they are wanting to get somewhere as fast as possible on their own clocks, i.e. regardless of how much time the trip takes from perspective of "couch-potatoes" in the map-frame.

This is the place to introduce constant proper-acceleration equations for proper-velocity $\vec{w}[\tau]$ and displacement $\vec{r}[\tau]$ in terms of the proper-acceleration 3-vector $\vec{\alpha}$ and the initial proper-velocity "at turnaround" wo given in our (3+1)D table, which at low speeds reduce to the familiar $\vec{w}_o + \vec{\alpha}\tau$ and $\vec{w}_o\tau + \frac{1}{2}\vec{\alpha}\tau^2$, respectively. In that limit also aging-factor $\gamma[\tau]$ goes to $1 + \frac{1}{2}(w/c)^2$ (equal to rest plus kinetic energy, divided by mc^2), and map-time $t[\tau]$ is basically traveler-time τ.

For unidirectional motion at any speed, the trajectory looks like $\vec{w}[\tau] = \tau_o \sinh[\tau/\tau_o]\vec{\alpha}$ and $\vec{r}[\tau] = \tau_o^2(\cosh[\tau/\tau_o] - 1)\vec{\alpha}$, where characteristic time τ_o becomes simply c/α (about a year for 1-gee $\simeq 1\ ly/y^2$ accelerations). These latter equations are quite useful e.g. for analyzing constant proper-accceleration roundtrips between two stars.

c. *map-based dynamics*

In order to track interactions (particularly of conserved quantities like energy $E = \gamma mc^2$ and momentum $\vec{p} = m\vec{w}$) between multiple traveling objects, we now must move beyond those traveler-point variables, which sadly are good for describing the motion of only one traveler at a time. This will lead to a work-energy theorem ($\delta E = \Sigma\vec{\xi}\cdot\delta\vec{x} = m\vec{\alpha}\cdot\delta\vec{x}$) familiar at any speed, and a map-based force-momentum theorem, namely $\Sigma\vec{f} = \delta\vec{p}/\delta t$, where each map-based force $\vec{f}$ equals $\vec{\xi}_{||\vec{w}} + \vec{\xi}_{\perp\vec{w}}/\gamma$ in terms of the corresponding frame-invariant felt-force $\vec{\xi}$ detectable e.g. by the traveler's cell-phone accelerometer. The latter, in turn, obey the frame-invariant felt force-acceleration relation $\Sigma\vec{\xi} = m\vec{\alpha}$.

An everyday example of this is the use of frame-invariant felt force $\vec{\xi} = \vec{f}_{||\vec{w}} + \gamma\vec{f}_{\perp\vec{w}}$ to show how the magnetic force on a positive charge moving at coordinate-speed v opposite to the current I in a neutral wire is, from the moving charge's (traveler-point) perspective, nothing more than the purely-electrostatic repulsion from a wire with charge-density $\lambda = \gamma vI/c^2$. Put another way, magnetic forces are a direct and everyday consequence of these "anyspeed" dynamical laws and the purely-electrostatic felt forces $\vec{\xi}$ that follow from Coulomb's law.

In fact, when in the "primed" traveler frame the felt force is electrostatic i.e. $\vec{\xi} = m\vec{\alpha} = q\vec{E}'$, then the total map-based force $\vec{f} = \vec{f}_E + \vec{f}_B$ can be broken (using SI units) into electrostatic $\vec{f}_E = q\vec{E}$ and magnetic $\vec{f}_B = q\vec{v}\times\vec{B}$ components. When expressed in terms of the electrostatic $\vec{E}'$ and magnetic $\vec{B}'$ fields in the traveler frame, these map-frame

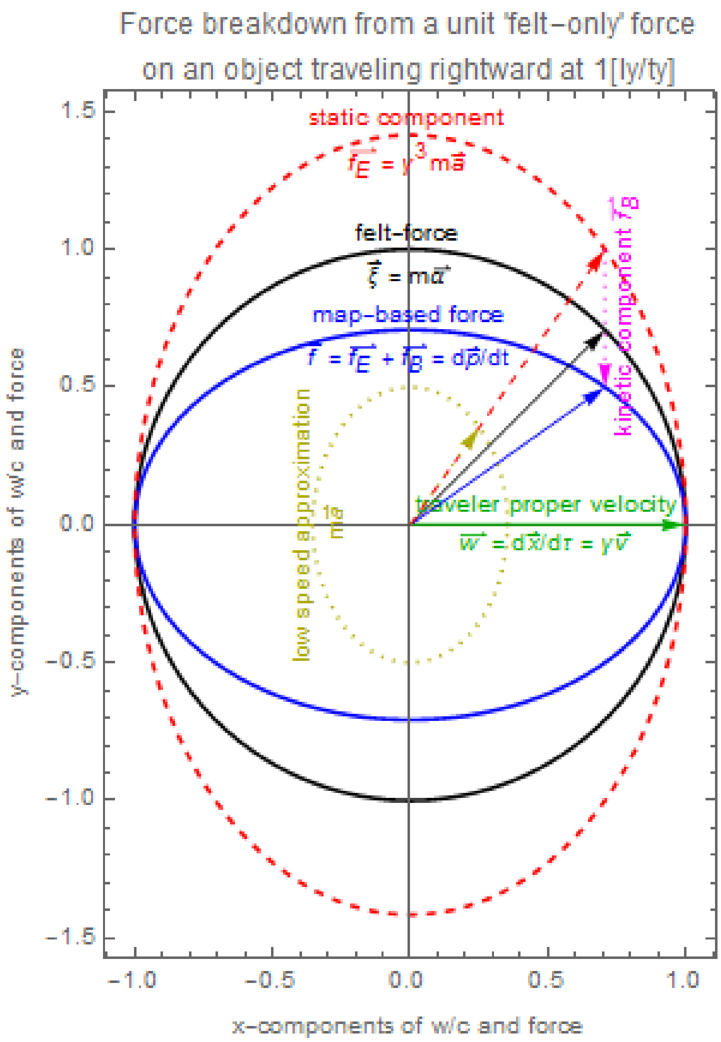

FIG. 5. Translating a "purely felt" force $\vec{\xi} = m\vec{\alpha}$ into a map-based sum $\vec{f} = \delta\vec{p}/\delta t$ of static (dashed) and kinetic (dotted) components.

components are[22] quite generally:

$$\vec{f}_E = q\vec{E}'_{||\vec{w}} + \gamma q\vec{E}'_{\perp\vec{w}} - q\vec{w} \times \vec{B}' \tag{A1}$$

and

$$\vec{f}_B = \left(\frac{1}{\gamma} - \gamma\right) q\vec{E}'_{\perp\vec{w}} + q\vec{w} \times \vec{B}'. \tag{A2}$$

If the force in the traveler frame is moreover "purely felt" i.e. $\vec{B}' = 0$, then the static component is also $\vec{f}_E = \gamma^3 m\vec{a}$ and therefore in the direction of coordinate-acceleration[17] $\vec{a} \equiv \delta\vec{v}/\delta t$. More generally, whenever a moving object accelerates itself with help from any "purely felt" net force $\vec{\xi}$ (like the thrust from a rocket engine cf. Fig. 5), the rate of momentum change $\vec{f} \equiv \delta\vec{p}/\delta t$ has a static component $\vec{f}_E = \gamma^3 m\vec{a} = \vec{\xi}_{||\vec{w}} + \gamma\vec{\xi}_{\perp\vec{w}}$ which is parallel to the observed coordinate acceleration $\vec{a} = \delta\vec{v}/\delta t$, and a corrective kinetic component $\vec{f}_B = (1/\gamma - \gamma)\vec{\xi}_{\perp\vec{w}}$ which bends the total map-based force $\vec{f} \equiv \delta\vec{p}/\delta t = \vec{f}_E + \vec{f}_B$ back toward the object's line of travel without increasing the object's speed $|\vec{w}| = |\gamma\vec{v}|$, as shown in Fig. 5.

The force-momentum and force-acceleration relations reduce to Newton's 2nd law ($\Sigma\vec{f} \simeq m\vec{\alpha} \simeq m\vec{a}$) in terms of coordinate-acceleration $\vec{a} \equiv \delta^2\vec{r}/\delta t^2$ at low speeds i.e. when $\gamma^3 \simeq 1$. For unidirectional motion, map-based forces $\vec{f}$ and frame-invariant felt forces $\vec{\xi}$ are equal, so that $\Sigma\vec{f} = \delta\vec{p}/\delta t = \Sigma\vec{\xi} = m\vec{\alpha}$ works at any speed. Momentum-conservation in the map-frame, of course, also gives rise to a general version of Newton's 3rd action-reaction law in flat spacetime, i.e. $\vec{f}_{AB} = -\vec{f}_{BA}$.

For the special cases of curved spacetimes (like that we live in here on earth) and for accelerated frames, this may also be a good place to discuss the utility of pretending that non-proper geometric forces (like gravity and centrifugal), which are invisible to our cell-phone accelerometers, are also proper forces for the purpose of predicting local trajectories. Worked examples of these include the link between differential-aging (through γmc^2) and potential energies U associated with: (i) Schwarzschild gravity i.e $-GMm/r$, (ii) centrifugal "gravity" i.e. $-\frac{1}{2}m\omega^2 r^2$, and (iii) a linearly accelerating spaceship i.e. $m\vec{\alpha} \cdot \delta\vec{x}$. Such geometric-force "potential energies" may be obtianed for time-independent metrics in the low-speed weak-field limit by setting $\delta t/\delta\tau \simeq 1/\sqrt{g_{00}}$, where g_{00} is the time-only metric coefficient.

Three-vector addition of relative proper-velocities may also be useful to mention here. Folks might enjoy this even less than they do the relative-velocity section in a standard Newtonian text, although it has the advantage that it works at any speed using a very similar construction. It also makes it pretty obvious why a collider is a way better investment than an accelerator, if you are trying to break the relative land speed record (in proper-velocity units, of course) for an electron, a proton, or a uranium nucleus. Worked examples of this include (i) the unidirectional relation where coordinate-velocities add while the aging-factors multiply to yield a relative proper velocity i.e. $\vec{w}_{AC} = \gamma_{AB}\gamma_{BC}(\vec{v}_{AB} + \vec{v}_{BC})$, and (ii) more general 3-vector proper-velocity addition problems using $\vec{w}_{AC} = \vec{w}_{AB^*} + \vec{w}_{B^*C}$ where $\vec{w}_{AB^*} = \vec{w}_{AB\perp\vec{w}_{BC}} + \gamma_{BC}\vec{w}_{AB||\vec{w}_{BC}}$ and $\vec{w}_{B^*C} = \gamma_{AB}\vec{w}_{BC}$, e.g. involving an enemy spaceship dropping out of hyperspace in the neighborhood of a starfleet ship orbiting a ringworld.

2. Symbols

In this section of the supplementary material, we provide a set of symbol definitions (Table III) designed to *extend*, rather than to replace, the standard set of variables used for intro-physics, so that they might *in addition* be used for applications at any speed. Our notation is also consistent with that adopted by Messerschmitt[17], where he invokes the phrase "denizen" as the map-based complement to the "traveler", and refers to the spacelike component of the acceleration 4-vector as "traveler's map acceleration" denoted by the 3-vector $\vec{b}$.

We are also using the greek letter $\vec{\xi}$ for forces as felt by the traveler, just as greek $\vec{\alpha}$ is the acceleration felt by the traveler, because it looks a bit like the lower case $\vec{f}$ which replaces it in both force-acceleration and work-energy relations for unidirectional motion, as well as in the Newtonain approximation. The appoach also allows us to put the approximate local use of "geometric forces", like gravity and centrifugal, into a modern integrative context.

3. Numerical examples

In this section of the supplementary material, we provide the result of some sample calculations in Table IV, to provide some high precision numerical examples that may be useful to developers and teachers in checking algorithms. It is also referred to in the main paper as a numerical illustration of the differential-equation solution's invariance relationships.

4. Sample acceleration problem

Although intended to include discussions of Figs. 2, 3 and 4, we illustrate with a discussion of only the latter here. A starfleet battlecruiser (black) traveling upward at 1.0 $[ly/ty]$, on seeing light from an enemy spaceship (gray) dropping out of hyperspace (dotted red circle) at constant speed, accelerates rightward at 1.0 $[ly/ty]$ in order to intercept. This allows it to dock with the enemy ship 1.0 traveler year later. It then reverses thrusters so as to begin a return to its starting trajectory.

1. Questions to ask, using only the "consensus" bookkeeper frame pictured to define simultaneity between spatially-separated events, might include:

TABLE III. Extended symbol set, with upper-case Latin used for 4-vectors and components, like $X^\mu \equiv \{X^t, \vec{X}\}$, and superposed arrows used to denote 3-vectors.

Name	Symbols	Comments
proper or traveler time	$\tau \equiv \sqrt{g_{\mu\nu} X^\mu X^\nu}/c$	frame-invariant time elapsed e.g. along a traveler's world line.
lightspeed constant	$c \equiv \sqrt{g_{\mu\nu} U^\mu U^\nu}$	frame-invariant magnitude of the velocity 4-vector U.
proper acceleration	$\alpha \equiv \sqrt{-g_{\mu\nu} A^\mu A^\nu}$	frame-invariant magnitude of acceleration 4-vector A and of the proper-acceleration 3-vector $\vec{\alpha}$.
map time	$t \equiv X^t/c$	scalar time parameter in the "bookkeeper metric" used to define simultaneity
speed of map time	$\frac{\delta t}{\delta \tau} \equiv \gamma \equiv U^t/c = \frac{E}{mc^2}$	scalar differential-aging (Lorentz) factor linked to kinetic & geometric-force potential energies
timelike acceleration	$\frac{\delta^2 t}{\delta \tau^2} \equiv A^t/c = \gamma \frac{P}{mc^2}$	scalar proper-time derivative of Lorentz-factor linked to work-energy theorem $\delta E = \Sigma \vec{\xi} \cdot \delta \vec{x}$, where proper felt-forces $\vec{\xi}$ obey $\Sigma \vec{\xi} = m\vec{\alpha}$ and $P \equiv \delta E/\delta t$
map displacement	$\vec{r} \equiv \vec{X}$	3-vector displacement in the "bookkeeper metric" used to define position
proper velocity	$\frac{\delta \vec{r}}{\delta \tau} \equiv \vec{w} \equiv \vec{U} = \frac{\vec{p}}{m} = \gamma \vec{v}$	3-vector map-distance traveled per unit traveler time linked to momentum $\vec{p} = m\vec{w}$. Here $\vec{v} \equiv \delta \vec{x}/\delta t$.
spacelike acceleration	$\frac{\delta^2 \vec{r}}{\delta \tau^2} \equiv \vec{A} = \frac{\delta \vec{p}}{\delta \tau} = \gamma \Sigma \vec{f}/m$	3-vector proper-velocity change per unit traveler time linked to map-based forces $\vec{f} = \vec{\xi}_{\vert\vert \vec{w}} + \vec{\xi}_{\perp \vec{w}}/\gamma$ through Newton's laws $\Sigma \vec{f} = \delta \vec{p}/\delta t$ and $\vec{f}_{AB} = -\vec{f}_{BA}$.

(a) How fast and in what direction was the enemy ship traveling, in map-distance per unit time on enemy-ship clocks?

(b) How many "gees" (net proper-force per unit mass) did the battlecruiser experience during the rendezvous maneuver?

(c) How long, in battleship and in map years, did the "ignition to docking" leg of the detour take? Note that we are not asking how long it took on enemy ship clocks. Why?

(d) Where (relative to an origin when first-light from the enemy ship arrived) did the docking take place?

(e) How much kinetic energy and momentum per unit mass did our battlecruiser gain during the speedup, from the map-frame perspective?

TABLE IV. Five examples illustrating the first-derivative (timelike) and second-derivative (spacelike) "metric hypoteneuse" relations via the match between the first two and last two columns. Here proper-acceleration α, plus starting proper-velocity w_o and elapsed traveler-time τ from turnaround are chosen randomly between 0 and 2 in units where $c = 1$. Intermediate velocity and acceleration "component predictions" are also provided, to make manual checks on the results easier. The values in such tables also agree if we allow space/time conversion constant c (AKA "lightspeed") to vary randomly.

c	α	w_o	τ	$w_{\|\|}$	$w_\perp$	γ	$\gamma\Sigma f_{\|\|}/m$	$\gamma\Sigma f_\perp/m$	$\gamma P/(mc^2)$	c_{test}	α_{test}
1	0.74499	1.32609	0.472665	0.357626	1.35723	1.72336	0.779977	0.132778	0.266428	1.	0.74499
1	1.92718	0.892756	1.35403	5.98691	2.95676	6.75171	10.8383	4.40094	11.5379	1.	1.92718
1	1.24773	0.925914	1.04885	1.64859	1.30402	2.32773	2.26679	0.806067	2.057	1.	1.24773
1	1.0874	0.924361	0.890118	1.10106	1.12014	1.862	1.54802	0.4686	1.19729	1.	1.0874
1	1.14245	1.6369	0.413079	0.484021	1.70017	2.03097	1.23076	0.310179	0.552972	1.	1.14245

(f) Just after ignition and just before the docking event, what were the apparent forces and rates of energy increase per unit mass to our battlecruiser from the map-frame perspective.

(g) Assuming that the battleship does a final rightward constant acceleration burn to return to its original trajectory, how much ship time is lost or gained as a consequence of the detour? How much map time is lost or gained, as well?

2. Questions that involve more than one map-frame might include:

   (a) Before the beginning of the chase, what was the velocity (magnitude and direction) of the enemy ship with respect to the battlecruiser.

   (b) Before the beginning of the chase, what was the velocity (magnitude and direction) of the battlecruiser with respect to the enemy ship.

Variations might involve different numbers, non-orthogonal starting velocities & accelerations, plus (even more challenging) thrust recommendations by shipboard computers for rendezvous given only data on the moving target. The multi-frame problems, like traditional special relativity problems involving multiple Lorentz transform frames, because of the need for multiple definitions of extended simultaneity seem to have more potential for confusion.

The key parameter for addressing all legs of the starfleet battleship's detour is the trajectory's characteristic time, given by $\tau_o = c/\alpha\sqrt{(\gamma_o + 1)/2} \simeq 1.09868[ty]$, where the differential-aging (Lorentz) factor at turnaround is $\gamma_o = \sqrt{1 + (w_o/c)^2} \simeq 1.41421$. Given this::

1. "One-map" or single bookkeeper metric questions, where it may be convenient to define generalized hyperbolic velocity angle or *rapidity* as $\eta \equiv \tau/\tau_o$, might be addressed as follows:

   (a) When the enemy velocity matches the battleship velocity of $w_B = \frac{1}{2}(1 + \cosh[\eta])\vec{w}_o + \tau_o \sinh[\eta]\vec{\alpha}$, we can infer that $w_E \simeq \sqrt{1.221812^2 + 1.14392^2} = 1.67372[ly/ty]$ at roughly $\tan^{-1}[1.22181/1.1439] \simeq 46.886^o$ counterclockwise from right;

   (b) The proper-acceleration given as $\alpha = 1[ly/y^2] \simeq 9.51287[m/s^2]$ amounts to (from units conversion) about 0.970043 (or nearly one) $[gee]$;

   (c) The docking segment was given to take $\Delta\tau_B$ of about one [battleship year], which translates to a map-time elapsed $\Delta t \simeq 1.58792[y]$ (map years), while time-elapsed between these events on enemy clocks involves a different map frame for defining simultaneity between spatially separated events;

   (d) As above using equations from the bottom row of Table II in the paper, the docking takes place at displacement $\{\Delta x, \Delta y\} \simeq \{0.535485, 1.07195\}[ly]$ from ignition at the origin;

   (e) *Per unit mass from the map-frame perspective* during the speed up, the total kinetic energy increase is $\Delta K/m = c^2\Delta\gamma \simeq 0.535485c^2 \simeq 4.8127 \times 10^{16}[J/kg]$ while the momentum increase is $\Delta p/m = \Delta w \simeq 0.673716c \simeq 2.01965 \times 10^8[m/s]$, suggesting that adventures at relativistic speeds can be energetically costly especially if you have to carry fuel with you in the process;

   (f) *Per unit mass from the map-frame perspective* on ignition the starting power output $\gamma P/m \equiv (1/m)\delta E/\delta\tau_B = \alpha^2\tau_o \sinh[\eta]$ is zero [W/kg] since at turnaround the force is perpendicular to velocity, while before docking $\gamma P/m \simeq 1.1439c^2$ per year i.e. about 3.26451×10^9 [W/kg], and the initial frame-variant force $\gamma\Sigma f/m = |\frac{1}{2}(\sinh[\eta]/\tau_o)\vec{w}_o + \cosh[\eta]\vec{\alpha}|$ is $1[ly/y^2] = 1.0\alpha$ while just before docking

$\gamma\Sigma f/m \simeq 1.51938c$ per year i.e. about $4.81792[m/s^2] = 0.506463\alpha$, consistent with the upper limit on frame-variant forces discussed in the paper;

(g) There are four equivalent "legs" associated with the full detour, so that remarkably the traveler time *saved* by the detour is $4.28781[ty] - 4[ty] \simeq 3.45371$ [traveler months], but there's a different story from the vantage point of couch potatoes awaiting your arrival, as I'm afraid that they'll see the ship arrive $6.35169[y] - 6.06388[y] \simeq 3.45371$ [map months] late!

2. Here we have to go beyond the 1-map approach to use of the "multi-map" addition-equation for 3-vector proper-velocities, of the form $\vec{w}_{AC} = \vec{w}_{AB^*} + \vec{w}_{B^*C}$ where the first term in the vector sum $\vec{w}_{AB^*} = \vec{w}_{AB\perp\vec{w}_{BC}} + \gamma_{\rm BC}\vec{w}_{AB||\vec{w}_{BC}}$ involves a change in metric from $B \to C$ while the second term $\vec{w}_{B^*C} = \gamma_{\rm AB}\vec{w}_{\rm BC}$ only involves a change in clock from $B \to A$. In both cases we let R stand for our reference-map (or consensus-metric) frame, B for the pre-chase battleship frame, and E for the enemy ship frame, and are therefore given $\vec{w}_{BR} = \{0, 1\}[ly/ty]$ and $\vec{w}_{ER} \simeq \{1.1439, 1.2218\}[ly/ty]$ to get:

   (a) that $\vec{w}_{EB} \simeq \{1.1439, -0.22185\}[ly/ty]$; and

   (b) that $\vec{w}_{BE} \simeq \{-1.1439, -0.22185\}[ly/ty]$.

Two puzzles that might be fun to explore in the followup are:

- From the result for (1g), under what conditions does the time saved on battleship clocks as a result of the detour equal the time lost on map-clocks at the (originally constant-speed) destination point?

- From the multi-frame velocity addition analysis in (2) which shows that from R's point of view the velocities of B with respect to E, and of E with respect to B, are both Wigner-rotated in the direction of R's relative motion, under what conditions in general is the proper velocity of A with respect to B "equal and opposite" to the proper velocity of B with respect to A?