# Making the most of "one-map" concepts first in teaching metric-smart mechanics

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(Dated: August 5, 2005)

Minor shifts in presentation, to wit: (i) being specific about which clock, (ii) a short review of frame dependences, and (iii) discussion of locally-useful "geometric forces", might offer two advantages to introductory physics students. First, the students will experience less cognitive dissonance when they encounter relativistic effects. Secondly, the map-based Newtonian tools that they spend so much time learning about may be extended to high speeds, non-inertial frames, and even (locally, of course) to curved-spacetime. Various ways to extend them in the limited time available are also reviewed. (Adapted from arXiv:physics/0110020)

**CONTENTS**



## I. INTRODUCTION

This note is motivated by a sense that those of us teaching introductory physics should be on the lookout for ways to economize on our use of concepts, particularly since this may be the last physics course for most of our students. Hence, for example, we show here how telling students to keep track of "which clock", when defining time intervals, allows them to explore consequences of the metric equation (and apply their acceleration equations to high speed problems) before they are burdened with the more abstract task of considering multiple relativistic inertial frames. The goal however is not to "add content", but instead with the minor tweaks each of us can work into teaching at the grass-roots level, to gradually provide more "bang per byte". In this context...

Relativistic environment dwellers in the earth's gravitational well often experience time differently than space. Practical challenges for them thus begin and end in frame-specific terms[1], even if coordinate-free insights help visualize the symmetry of the problem[2]. Hence problem solvers in such environments can benefit from Newton-like laws [with units] based in their own frame of reference, that model nature over as wide a range of (location and velocity) conditions as possible[3].

Just as some fluid-flow tools that we offer to introductory students don't apply under supersonic conditions, so the tools for thinking about motion fail in fundamental ways at relativistic speeds and in curved spacetime. We discuss here ways to hedge our bets with the tools used to describe motion – looking at the pros and cons of some changes in emphasis that can minimize the pain of deeper understanding later. These changes might also open the door to wider application of frame-based tools in projects involving high speeds and/or extreme spacetime curvature. For example, few children know that relativity enhances their options for long range travel in one lifetime, instead of limiting their travel to places within 100 light years of their place of birth. Likewise, describing the world land speed record for particle acceleration in terms of "map distance per unit traveler time" could augment public interest in accelerators if particle speeds are expressed in units of lightyears per traveler year, the current land speed record (e.g. for 50 GeV electrons) being on the order of $100,000$.

Access to improved tools can be facilitated by avoiding three assumptions implicit in the classical worldview: (i) global time, (ii) frame-invariant dynamics, and (iii) limitation to inertial frames. These assumptions generate cognitive dissonance when one later considers either high speed motion or modern views of gravity. Their elimination does not require major changes. It further facilitates the extension of familiar Newtonian tools to problems involving acceleration at high speeds, and (with help from the equivalence principle) to arbitrary coordinate systems as well.

## II. DISSONANCE ONE

"Time is the same for everyone." This is the implicit assumption of global time, against which most spoken languages have little built in protection.

### A. The Specified-Clock Fix

Make a habit of specifying the clock used to measure time $t$, as well as the coordinate system yardsticks used to measure position $x$. With time specified as measured on map-clocks (i.e. a set of synchronized clocks connected to the yardsticks with respect to which distance is measured), the usual kinematic equations of introductory physics need not be changed, even if their precise meaning (at least in the case of equations describing relative motion) is different.

### B. Consequences of Specified-Clocks

When $t$ represents time on "map-clocks", the usual definitions of velocity and acceleration (as derivatives of map-position with respect to $t$) define what relativists sometimes call "coordinate" velocity and acceleration, respectively[2]. These definitions are useful at any speed, although they become less global when spacetime is curved. If the integrals of constant acceleration, like $\Delta v = a\Delta t$ and $\Delta(v^2) = 2a\Delta x$, represent relations between map-time and map-position, they follow simply from these definitions and hence are also good at any speed. The caveat, of course, is that holding coordinate acceleration finite and constant is physically awkward at high speeds, and impossible to do forever. This diminishes the usefulness of these equations in the high speed case[4], but not their correctness. Thus being specific about clocks yields classical definitions of velocity and acceleration, plus familiar equations of constant coordinate-acceleration, that work even at relativistic speeds.

The kinematical equations for *relative motion* (e.g. $x' \cong x + v_o t$ and its time derivatives) now refer only to yardsticks and clocks in a single reference frame. Hence they only tell us about "map-frame" observations. While they are correct in these terms, the prediction of prime-frame perspectives is only valid at low speed. Since multiple inertial frames are a challenge in and of themselves[5], describing the relative motion equations as low-speed approximations may be the best choice in an introductory course. Still, much can be done from the perspective of a single inertial frame to give quantitative insight into motion at high speeds.

If for example traveler-time $\tau$ is introduced as discussed below, the unidirectional velocity-addition rule $v_{13} = v_{12} + v_{23}$ can be simply multiplied by $\gamma \equiv dt/d\tau$ values so as to work at any speed, i.e. the change in displacement of object 1 in terms of object 3's yardsticks and simultaneity, per unit time on the clocks of system 1, is $\gamma_{13}v_{13} = \gamma_{12}\gamma_{23}(v_{12}+v_{23})$, where $v_{12}$ is the coordinate-velocity of object 1 from object 2's perspective, etc. This equation allows students to instantly account e.g. for the ability of a super-collider to pull off a 100,000-fold increase in collision energy with two $\gamma = 10^5$ particles, in comparison to collisions involving the same two particles against a fixed target. This follows from above with the earth as object 2, since in that case $v_{12} = v_{23} \simeq c \simeq v_{13}$.

Of course the cost in class time spent on multi-frame position and velocity transforms must be weighed against benefit. Even if no time is available for relativity, with only subtle change introductory physics courses can become a natural venue for amending the assumption of "global-time".

#### *1 Extension: Fast Airtracks*

Rather than simply "being told" about relativity, students might benefit from time spent first pondering how to explain data on traveler/map time differences, in the spirit of Piaget and modeling workshop[6]. For example, a simulated airtrack experiment, with adjustable kinetic-energy source (e.g. a spring) and two gliders that stick on collision, is easy to design that uses only local measurements to avoid communication-lag errors at high speed. Work with such a setup can give students data for "experimental discovery" in their own terms of phenomena underlying the metric equation.

Discoveries that might be made from such "data", as shown in Fig. 1, include the metric equation and the units-conversion between distance and time i.e. "light-speed" $c$. From spring-compression and post-collision data, relativistic expressions for conserved quantities (i.e. kinetic energy $K = (dt/d\tau - 1)mc^2$ and momentum $p = mdx/d\tau$) might be deduced as well.[7] Of course, a particular student's exploration might uncover, instead of the metric equation, the mathematically-equivalent relation that holds constant the sum of squares of a traveling object's "coordinate speeds" through space ($dx/dt$) and time ($cd\tau/dt$). Note that all of these quantities are simple functions of the gate separation ($dx$) measured in the lab, and the time between gates as measured by lab ($dt$) and glider ($d\tau$) clocks.

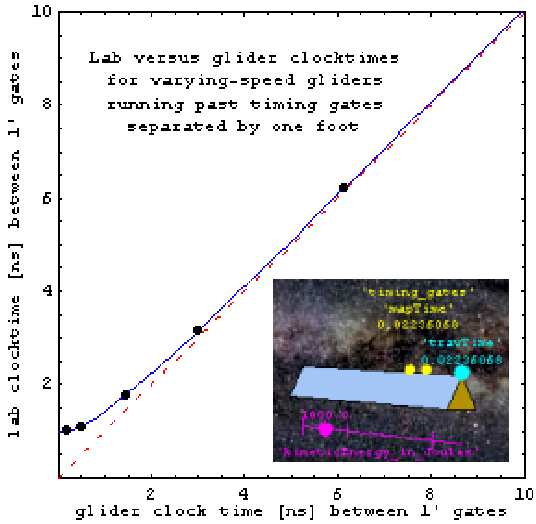


FIG. 1: One way to plot data from a web-interactive air-track experiment.

*2 Extension: One-Map Two-Clocks*

The conceptual distinction between stationary and moving clocks opens the door for students to discover, and apply, Minkowski's spacetime version of Pythagoras' theorem[8] $(cd\tau)^2 = (cdt)^2 - (dx)^2$, i.e. the metric equation, as an equation for time $\tau$ on traveler clocks in context of coordinates referenced to a single map frame[9]. The metric equation connects (i) traveler and map time intervals, (ii) coordinate velocity, $v \equiv dx/dt$, with proper velocity[10], $w \equiv dx/d\tau = \gamma v$, and (iii) special to general relativity as well. The metric equation also nicely serves up Lorentz transforms[11] when challenges (like length contraction) come up that involve multiple map-sets of yardsticks and synchronized clocks (i.e. more than one definition of simultaneity).

One consequence of recognizing that proper-time in the metric equation represents a "local invariant" of interest, rather than a whole second map-frame of yardsticks and synchronized clocks, is that time-dilation problems become easy to visualize in the context of a single map-based definition of simultaneity. Fig. 2 illustrates by comparing a constant-speed relativistic twin adventure with one involving constant proper-acceleration (about which more later).

Another consequence of such an introduction to clock behavior at high speed is that an upper limit on coordinate-velocity $v$ may seem natural to students since the metric equation connects it to the lack of such a limit on proper-velocity $w$ (and momentum). One would hardly expect this latter quantity (map-distance traveled per unit traveler-time) to exceed infinity for a real world traveler. The absence of an upper bound also makes a proper-velocity of one [lightyear per traveler-year] at $v = c/\sqrt{2}$ a natural "scale speed" for the transition from sub-relativistic to relativistic regimes. Einstein's gamma factor is also naturally defined as a speed of map-time per unit traveler-time given simultaneity defined by the map, i.e. $\gamma \equiv dt/d\tau$. With such definitions the metric equation easily yields the familiar relations, $\gamma = 1/\sqrt{1-(v/c)^2} = w/v$.

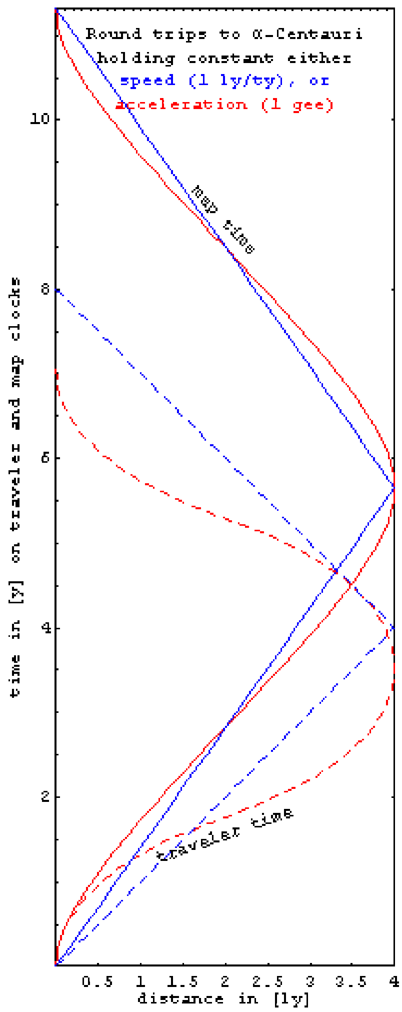


FIG. 2: "Single-frame" view of relativistic-twin adventures.

Although the metric equation can "serve up" length-contraction in terms of the proper-length of an object in flat spacetime, the concept of proper-length itself (as distinct from time-elapsed on a traveler's clock) is delocalized and requires its own definition of simultaneity, and rest-frame map of yardsticks and clocks. Put another way, on an x-ct diagram time-dilation is about the separation between only two events, while length-contraction involves three events and in effect two map-frames. Just as relative motion is a level of complexity up from single-frame Newtonian kinematics, so the multiple-frames of length-contraction and Lorentz transforms[5] are a level up from anyspeed kinematics that involve only one map-frame and one definition of simultaneity.

TABLE I: Quantities useful in the study of motion, all of whose values depend on one's choice of reference frame except for the three in **boldface**, whose values are frame-invariant.

| | | | |
|---|---|---|---|
| *Kinematic parameters* | | | |
| Coord-incr's: | map-distance $\Delta x[ly]$ | map-time $\Delta t[y]$ | **traveler-time** $\Delta\tau[ty]$ |
| $1^{st}$ deriv's: | coord-velocity $v \equiv \frac{dx}{dt}[\frac{ly}{y}]$ | map-time speed $\gamma \equiv \frac{dt}{d\tau}[\frac{y}{ty}]$ | proper-vel. $w \equiv \frac{dx}{d\tau}[\frac{ly}{ty}]$ |
| $2^{nd}$ deriv's: | coord-accel. $a \equiv \frac{dv}{dt}[\frac{ly}{y^2}]$ | | **proper-accel.** $\alpha[\frac{tly}{ty^2}]$ |
| *Dynamical parameters* | | | |
| Integrals: | momentum $p \equiv mw$ | energy $E \equiv \gamma mc^2$ | |
| $1^{st}$ deriv's: | force $\Sigma F \equiv \frac{dp}{dt}$ | power $P \equiv \frac{dE}{dt}$ | **proper-force** $F_o \equiv m\alpha$ |

*3 Drawbacks: Irrelativity*

The downside of specifying clocks, of course, is that global time is more deeply-rooted and simpler than clock-specific time. Tradition also treats magnetism and gravity as non-relativistic add-ons to a classical world. The simplicity defense may not hold up, therefore, if our present understanding of *all these phenomena* grows from a metric equation that looks like Pythagoras' theorem with a minus sign. Thus perhaps *only tradition* denies that relativistic effects color our lives everyday.

## III. DISSONANCE TWO

"Coordinate acceleration and force are frame-invariant." Informal polls at one state[12] and one national[13] AAPT meeting suggest that physics teachers often presume that the rate of change of conserved quantities (energy as well as momentum) is frame-invariant, even though the expression for power as a product of force and velocity makes this clearly incorrect at even the lowest speed.

### A. The Subjective-Dynamics Fix

Address the issue of observer-dependence when dynamical quantities are first introduced, for example with help from a list like that in Table I. Point out that the value of the conserved dynamical quantities momentum and energy, along with their time derivatives force and power, depends on the frame of reference even though force is nearly frame-independent at low speeds. This by itself, at the foundation, might be adequate to keep Newton's implicit assumption of global force and acceleration from becoming an *unconscious* part of student perceptions.

You might be tempted to further note that coordinate acceleration $a = dv/dt$ is not necessarily the acceleration experienced in the proper frame of a traveler. The former is impossible to hold finite and constant indefinitely, while proper acceleration $\alpha$ (the magnitude of the acceleration 4-vector) is both frame-invariant and possible to hold constant[14]. This is what Newton had guessed (incorrectly) to be the case for coordinate acceleration, by a perfectly reasonable application of Occam's razor given the observational data that he had to work with at the time[15,16].

Quantifying these accelerations at arbitrary speed is simple in the unidirectional case[17], where $\alpha = \gamma^3 a$. An invariant proper-force $F_o = m\alpha$, useful for "one-frame derivations" like that in Appendix A, may also be defined in this context[18]. Quantifying the conserved dynamical quantities further requires relativistic expressions for momentum $p = mw$ and energy $E = \gamma mc^2$. Their derivatives with respect to map-time then become the familiar net frame-variant force $\Sigma F = mdw/dt$ and power $dE/dt = \Sigma F \bullet v$.

These quantities most directly represent the way that momentum and energy associated with a traveling object is changing, from the perspective of a map-frame observer. Proper-time derivatives of momentum and energy are of course easier to transform from frame to frame (they form a 4-vector), but they also represent a perspective intermediate between that of the map-frame observer and the traveling object. The focus here is on local implementation of the insights in terms of map-frame observer and traveling-object experiences. Hence references here to coordinate-free insights that underlie the conclusions (e.g. to the four-vector nature of certain quantities) are for instructor reference, but of limited use to students wanting a sense of anyspeed motion only in concrete terms.

Perhaps coincidentally in the unidirectional motion case, the net frame-variant force $\Sigma F$ and the proper-force $F_o$ are equal. In flat (3+1D) space-time, proper acceleration and proper force behave like a 3-component scalar (i.e. no time component) within the "local space-time coordinate system" of an accelerated object[2], while the corresponding frame-variant forces differ and may be accompanied by frame-variant energy changes. The potentials that give rise to such frame-variant forces inevitably have both time-like (e.g. electrostatic) and space-like (e.g. magnetic) components[8]. In curved space-time and/or rotating coordinate systems, the proper-acceleration and proper-force retains its three-component form only from instant to instant, within the traveling object's "proper reference frame"[2].

### B. Consequences of Subjective-Dynamics

A brief but explicit discussion in the intro-physics classroom of which quantities depend on one's frame of reference might nip a number of misconceptions in the bud. This is especially important since relativity has uncovered the fact that many more quantities are dependent on one's choice of reference frame than Newton expected. For some quantities (elapsed times, object length, mass[19], traveler velocity[20], force, and acceleration[18]) there is a "minimally variant" or proper form which can simplify discussion across frames. As summarized in Table I, relativity also points out the existence of a number of quantities which remain frame-invariant even in curved spacetime. Applications for some of these are discussed in the extensions below.

### 1 *Extension: Proper Acceleration*

Recognizing the distinction between coordinate and proper acceleration, students also gain (from the metric equation) simple 1D equations for describing constant proper-acceleration. In terms of coordinate integrals like those for constant coordinate-acceleration above, there are three equations instead of two[21] because the metric equation relates three coordinates: map-time $t$, map-position $x$, and traveler-time $\tau$. The integrals are $\Delta w = \alpha\Delta t$, $c^2\Delta\gamma = \alpha\Delta x$, and $c\Delta\eta = \alpha\Delta\tau$ where the "hyperbolic velocity angle"[18] or rapidity $\eta \equiv \tanh^{-1}[v/c]$. One can also give students a set of map-based Newton-like laws good in (3+1)D flat spacetime, with only a modest amount of added complication[22].

Since all of these quantities may be defined in the context of a single map-frame of yardsticks and synchronized clocks, their values as a function of position may all be graphed as a function of position (in x, y and z) at once. This is illustrated for the case of a uni-directional accelerated twin-adventure in Figure 3.

In context of a single map-frame in (3+1)D, Newton's second law for flat spacetime might be written as

$$\Sigma\overleftarrow{F} = m\frac{\alpha}{\gamma_t}\overleftarrow{i}_l \overset{v_\perp \ll c}{=} m\overleftarrow{\alpha} \overset{v \ll c}{=} m\overleftarrow{a} \tag{1}$$

where column three-vectors are denoted by left-pointing arrows, and unit vectors are denoted by the letter $i$. As usual $v$ and $a$ denote coordinate-velocity and coordinate-acceleration, while $\alpha$ denotes proper-acceleration. Vector components transverse and longitudinal to the frame-variant force direction are denoted by subscripts $t$ and $l$, respectively, while components perpendicular and parallel to the frame-invariant proper-acceleration three-vector are denoted by subscripts $\perp$ and $\|$. Also $\gamma_t \equiv \sqrt{1+(w_t/c)^2}$, and the unit vector longitudinal to the frame-variant force is related to unit vectors perpendicular and parallel to the proper-acceleration by

$$\overleftarrow{i}_l = \gamma_t\gamma_\perp\left(\frac{1}{\gamma_\perp^2}\overleftarrow{i}_\| + \frac{v_\perp}{c}\frac{v_\|}{c}\overleftarrow{i}_\perp\right), \tag{2}$$

where $\gamma_\perp \equiv 1/\sqrt{1-(v_\perp/c)^2}$. Hence the frame-variant force direction differs from that of the invariant proper acceleration only when the velocity of the object being effected is neither perpendicular to, nor parallel to, the proper acceleration.

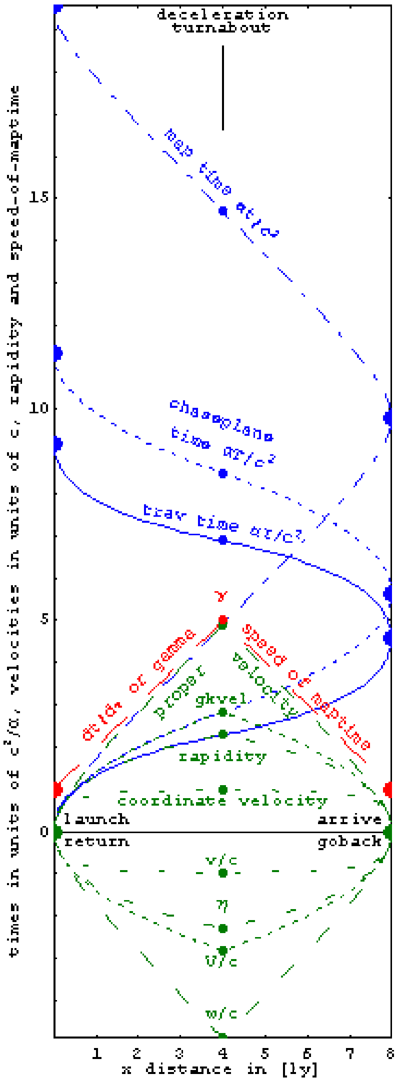


FIG. 3: "Single-frame" view of variables useful in describing a round-trip involving constant proper-acceleration.

Just as the invariant proper-time can be usefully plotted in context of a single-frame definition of position and simultaneity (cf. Figs 1, 2, and 3), so the invariant "felt" or proper-force might be usefully plotted along with more familiar frame-variant dynamical quantities. The figure in Appendix A on "one-frame Biot-Savart" provides an example of this.

### 2 *Extension: Galileo's Chase-Plane*

Relativistic equations for constant proper acceleration work poorly at low speeds because of roundoff error in calculating the difference between squares, while Galileo's equations[23] (which students spend so much time learning) are elegant in their simplicity. Perhaps it is good news then that by going from "one-map two-clock" descriptions of motion *ala* the metric equation to one-map and three-clocks, Galileo's equations can be shown to apply to constant proper-acceleration at any speed.

Specifically, in terms of time T on the clocks of a suitably-motivated "chase-plane", one can show[24] that the Galilean-kinematic velocity $V \equiv dx/dT$ (here x and simultaneity are still defined by the map-frame) of a traveler undergoing constant proper-acceleration $\alpha$ obeys $\Delta(V^2) = 2\alpha\Delta x$, $\Delta V = \alpha\Delta T$, etc. This familiar and simple time evolution seamlessly bridges the gap to low speeds since $v < V < w$, and $w \to v$ for $v \ll c$. It also predicts the experience of relativistic observers using the classic equations since, for example, at any speed $\gamma = 1 + \frac{1}{2}(V/c)^2$ and proper velocity $w = V\sqrt{1 + \frac{1}{4}(V/c)^2}$.

#### *3 Extension: One-Frame Magnetism*

The non-parallel relationship between frame-variant force and proper acceleration is manifest in our everyday life as a need to recognize both electrostatic and magnetic components to the Coulomb force between moving electric charges. This for example makes possible single-frame (one-map two-clock) derivations (cf. Appendix A) of the Lorentz Law and Biot-Savart in (3+1)D.

#### *4 Drawbacks: Inter-Scale Tension*

One downside of discussing the fact that observed forces depend on one's frame of motion (in flat space-time) and one's location (in curved spacetime) is the elegant simplicity (and for most engineers, the elegant practicality as well) of "global" force and acceleration in the Newtonian worldview. Refinement of prose to break this news to students in context (the "detail work" of content modernization) may therefore require a long period of experimentation and refinement. Even then, true global perspectives on frame invariance (and its absence) will likely remain a corollary rather than a pillar of introductory dynamics.

## IV. DISSONANCE THREE

"Newton's laws work only in unaccelerated frames." This assertion is often echoed in classes whose first example of a force is the affine-connection force[2] gravity, which like centrifugal force arises only if one chooses a "locally non-inertial" coordinate system. Of course this would be fine for historical reasons, if the equivalence principle hadn't elegantly shown that Newton's laws are useful *locally* in any frame[25]. This is potentially motivational news for those first struggling to understand how Newton's laws work, and perhaps worth sharing from the start.

### A. The Equivalence Fix

Tell students that Newton's laws work *locally* in any frame, but that in non-inertial frames they work only if one takes into account "geometric" (affine-connection) forces (e.g. centrifugal or gravity) that act on every ounce of one's being. Geometric forces also have the property that they may be made to vanish at any point in space and time, by choosing a "locally inertial" coordinate system. Some examples are given below. Of course non-local effects, like tides and coriolis forces, may not be possible to eliminate by one's choice of frame[2].

*Example 1*: On traveling around a curve of radius $r$ in a car at speed $v$, note that a weight suspended by a string from the rear view mirror accelerates away from the center of the turn. In the non-inertial frame of the moving car, this is instinctively seen as the consequence of a "centrifugal force". A more careful look shows that this force seems to act on every ounce of the object's being. For example, the resulting acceleration is to first order independent of object mass (equal to $v^2/r$), and it does not push or pull just on one side or the other. Secondly, the explanation is only useful locally. If an object is allowed to travel too far under this geometric force (e.g. more than 30 cm when going around a 30 m radius curve) complications arise in its motion not expected from a simple radially-outward force. Finally, note that this force vanishes if one observes events from the "locally-inertial" frame of a pedestrian standing by the side of the road. Free objects in the car are simply trying to move in a straight line in the absence of any force at all.

*Example 2*: When standing near the surface of the earth, note that when you drop an object, it falls. This is instinctively seen as a "gravity force". On closer inspection, this force acts on every ounce of an object's being in that it gives rise to an acceleration ($g$) that is independent of mass. No single part of a homogeneous object is pulled or pushed preferentially. Note also that this force vanishes if one observes events from the "locally-inertial" frame of a person falling with the object. To its falling companion, the object is simply trying to move in a straight line, in the absence of any force at all.

Denying the usefulness or reality of either of these forces denies the utility of the equivalence principle itself. The question is not if these forces exist. One's sense of being forced to the side of vehicles as they round curves is as real as one's sense of being forced to the ground when a chair leg breaks. Rather, we might better be asking: What is the range of positions and times over which such "geometric forces" can be seen to govern motion in their non-inertial setting, while remaining consistent with a set of Newton-like rules.

### B. Consequences of Equivalence

Students are clued into Einstein's integrative conceptual framework for putting Newton's Laws to work (locally) from any frame of reference. It even works in curved spacetime. They are also taught to keep an eye out for geometric forces, recognizable by virtue of the mass-independent accelerations that they cause. One might for example mention these concepts when discussing rotational motion, and then at the end of the section on mechanics ask if any of the other forces discussed (e.g. normal, friction, gravity, spring, or rope) are candidates for a geometric origin.

Equivalence also provides a segway into discussion of: (i) the anyspeed carousel and extreme gravity extension (below) of equations found in all intro-physics textbooks, (ii) non-local forces like tidal and coriolis which also arise in non-inertial frames, and (iii) other results of general relativity including the subjectivity of earth-based NIST clocks as manifest in global positioning system applications[26], and extreme environments like those discussed in Taylor and Wheeler's most recent text[27].

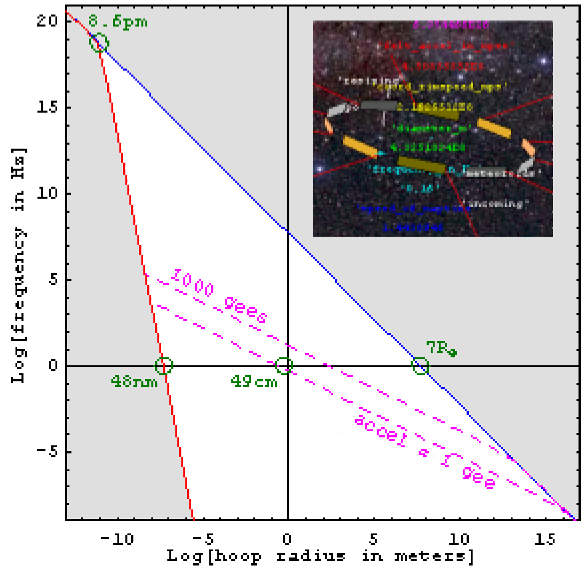


FIG. 4: Upper and lower limits on hoop radius as a function of rotation frequency. Insert shows one way length contraction might manifest at high speeds in a web-based artificial-gravity simulator.

### 1 *Extension: Anyspeed Carousels*

Consider a set of fiducial (map-frame) observers who find themselves rotating at angular velocity $\Omega$ along with a set of yardsticks arrayed around the circumference of a circle of radius $r$. In this case, the metric tensor for $x^0 = ct = ct_o\sqrt{1-(\frac{\Omega r}{c})^2}$, $x^1 = r = r_o$, $x^2 = \phi = \phi_o + \Omega t$, and $x^3 = z = z_o$ becomes

$$g_{\mu\nu} = \begin{pmatrix} -\left[1-\left(\frac{\Omega r}{c}\right)^2\right] & 0 & \frac{\Omega r}{c} r & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\Omega r}{c} r & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

As shown by Cook[28], the spatial metric defining local radar distance then assumes the decidedly non-Euclidean form

$$(d\ell)^2 = (dr)^2 + \frac{1}{1-\left(\frac{\Omega r}{c}\right)^2}(rd\phi)^2 + (dz)^2 \quad (4)$$

Thus distance is the same in the radial direction as in a stationary frame, while circumferential yardsticks are length-contracted in the azimuthal direction, increasing local radar distance as appropriate for an azimuthal velocity of $\Omega r$ and illustrating contraction-effects within one (azimuthal ring) frame.

To find the geometric forces, we calculate the affine-connection and resulting geodesic equation in terms of coordinates in this rotating frame. Thus free objects experience a radial acceleration of the form

$$\frac{d^2 r}{d\tau^2} = r\left(\gamma\Omega - \frac{d\phi}{d\tau}\right)^2, \quad (5)$$

where $\gamma$ is as usual $dt/d\tau$. As any car passenger can attest, accelerations like this are experienced as forces that act on every ounce of one's being. For observers at fixed $\phi$, this "affine-connection force" is a relativistic version of the familiar centrifugal force felt as one goes around curves in a car. This treatment of the problem goes further even for low-speed application. For observers with fixed $r$ and changing $\phi$ e.g. for azimuthal track runners in a space station with artificial gravity, the above equation predicts a change in their "centrifugal weight" depending on how fast and in which direction they run. In particular by running quickly enough in a direction opposite to the satellite's rotation, they can make themselves weightless.

This extension might be put to use in an intro-physics classroom as one of several "extreme physics" vignettes, designed to bridge the gap between an everyday physics topic and less easily-accessible but fascinating aspects of the world around. As illustrated in Fig. 4, the "artificial gravity" associated with hoops rotating at everyday rates (on the order of a cycle per second) makes contact with: (a) relativistic effects as the hoop diameter increases to several tens of earth diameters (yielding a excellent way to visualize length contraction from the perspective of a single frame), and (b) angular momentum quantization as the hoop diameter decreases to below a tenth of a micron (e.g. the size of a large virus molecule). Web simulators allowing students to "take data" on such systems might facilitate a visceral appreciation for these aspects of nature, long before any given student is ready to learn the physics needed for a quantitative understanding.

*2 Extension: Extreme Gravity*

The metric tensor for $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$, here in "far coordinates", becomes

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{2GM}{c^2 r}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{c^2 r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin[\theta]^2 \end{pmatrix}. \quad (6)$$

Although this space-time curvature only changes the metric coefficients by about a part per billion at the surface of the earth, it beautifully explains much of what we experience about gravity today, and more.

Of course, the equation only applies exterior to a spherically symmetric mass M. For objects whose mass lies within the *event horizon* radius predicted by this metric, at $r = \frac{2GM}{c^2}$, the metric also only applies exterior to the event horizon as well. Cook[28] shows (strangely enough) that for both "shell frame" (r constant) and "rain frame" (free falling from infinity) observers, the local physical metric (i.e. Pythagoras' theorem) remains Euclidean, at least outside the event horizon. This in spite of the fact that the two are obviously in different states of acceleration.

Calculating far-coordinate affine-connection terms, and the resulting geodesic equation, predicts a radial acceleration for stationary objects of the form

$$\frac{d^2 r}{d\tau^2} = \frac{GM}{r^3}\left(r - \frac{2GM}{c^2}\right). \quad (7)$$

This acceleration in far-coordinates thus for example goes away (because of the apparent slowing down of time) at the event horizon, but cleanly reduces to Newton's gravity law for $r \gg 2GM/c^2$.

*3 Drawbacks: Life on a Shell*

Disadvantages of discussing the local validity of Newton's laws, as distinct from their correctness only in unaccelerated frames, are at least twofold. To begin with, the concept of "local validity" is rather sophisticated. Admittedly we have many concrete examples, like the local validity of the uniform gravitational field approximation at the earth's surface, and its inability to deal with less local phenomena such as orbits and lunar tides. A deeper problem is the difficulty of explaining what an inertial (e.g. rain) frame is, if stationary frames (e.g. sitting down on a stationary earth) can be non-inertial. It is simpler (or at least more traditional) to introduce inertial frames by referring to their uniform motion relative to some inertial standard, rather than by referring to the absence of a felt "geometric" acceleration acting on every ounce of mass in one's corner of the world.

## V. CONCLUSIONS

Measurement of time and mass in meters provides relativistic insight into global symmetries, and the relation between 4-vectors in coordinate-independent form. However, frame-specific Newton-like laws with separate units for length, time and mass are perhaps still crucial to the inhabitants of any particular world, as an interface to local physical processes.

We point out some advantages of the fact that Newton's laws, written in context of a map-frame of choice, have considerable potential beyond their classical applications. By avoiding the implicit assumptions of (i) global time, (ii) frame-invariant dynamics, and (iii) limitation to inertial frames, introductory students can be better prepared for an intuitive understanding of relativistic environments, as well as for getting the most out of the laws themselves. The real challenge, however, which lies ahead is to develop course-specific lesson plans, and *spacetime concept inventories*[5] so that we might determine the effectiveness of these metric-based modifications (separately for physics and non-physics majors) in improving post-class student understanding.

## ACKNOWLEDGMENTS

Several decades of energetic work by E. F. Taylor, on pedagogically-informed content-modernization, has offered key inspiration.

## REFERENCES

[1] J. S. Bell, in *The speakable and unspeakable in quantum mechanics* (Cambridge University Press, 1987).

[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).

[3] N. Ashby, *Relativity in the future of engineering*, IEEE Trans. on Inst. and Meas. **34**(4), 505 (1994).

[4] A. P. French, *Special relativity*, The M.I.T. Introductory Physics Series (W. W. Norton, New York, 1968).

[5] R. E. Scherr, P. S. Schaffer, and S. Vokos, *Student understanding of time in special relativity: Simultaneity reference frames*, Am. J. Phys. Suppl. **69**(7), S24 (2001).

[6] D. Hestenes, *Modeling games in the Newtonian world*, Am. J. Phys. **60**, 732 (1992).

[7] P. Fraundorf, *An experience model for anyspeed motion* (2001), arXiv:physics/0109030.

[8] H. Minkowski, in *The principle of relativity - A collection*, edited by translated by W. Perrett and G. B. J. with annotations by A. Sommerfeld (Methuen and Company, London, 1923), delivered at the 80th Assembly of German Natural Scientists and Physicians in Cologne - 1908.

[9] E. F. Taylor, *The boundaries of nature, special and general relativity and quantum mechanics: A second course in physics*, Am. J. Phys. **66**, 369 (1998), the Oersted Medal lecture.
[10] Sears and Brehme, *Introduction to the theory of relativity* (Addison-Wesley, New York, 1968).
[11] T. Moore, *Six Ideas that Shaped Physics* (McGraw-Hill, 1998).
[12] (1997), the Illinois State AAPT Fall Meeting in Godfrey IL.
[13] (1998), the AAPT Winter Meeting in New Orleans.
[14] C. Lagoute and E. Davoust, *The interstellar traveler*, Am. J. Phys. **63**, 221 (1995).
[15] I. Newton, *Mathematical principles of natural philosophy - 1686* (U. California Press, Berkeley, 1946), translated by Motte in 1729, revised by Cajori in 1946.
[16] S. Chandrasekhar, *Newton's Principia for the common reader* (Clarendon Press, Oxford, 1995).
[17] A. J. Mallinckrodt, *What happens when at > c?* (1999), presented at the Summer 1999 AAPT meeting in San Antonio, Texas.
[18] E. Taylor and J. A. Wheeler, *Spacetime physics* (W. H. Freeman, San Francisco, 1963), 1st ed., contains some material not found in the 2nd edition.
[19] C. G. Adler, *Does mass really depend on velocity, dad?*, Am. J. Phys. **55**, 739 (1987).
[20] W. A. Shurcliff, *Special relativity: The central ideas* (1996), 19 Appleton St, Cambridge MA 02138.
[21] P. Fraundorf, *Teaching newton with anticipation...* (1997), arXiv:physics/9710013.
[22] P. Fraundorf, *Some minimally-variant map-based rules of motion at any speed* (1997), arXiv:physics/9704018.
[23] G. Galilei, *Dialog concerning the two chief world systems - 1629* (U. California Press, Berkeley, 1962), translated by Stillman Drake.
[24] P. Fraundorf, *Non-coordinate time/velocity pairs in special relativity* (1996), arXiv:gr-gc/9607038.
[25] A. Einstein, *The meaning of relativity* (Princeton Univ. Press, Princeton, NJ, 1950), 3rd ed.
[26] N. Ashby, in *Gravitation and Relativity at the turn of the Millenium*, edited by N. Dadhich and J. V. Narlikar (Inter-University Centre for Astronomy and Astrophysics, Pune India, 1998), pp. 231–258.
[27] E. Taylor and J. A. Wheeler, *Exploring black holes* (Addison Wesley Longman, 2001), 1st ed.
[28] R. J. Cook, *Physical time and physical space in general relativity*, Am. J. Phys. **72**, 214 (2004).

## APPENDIX A: ONE-FRAME BIOT-SAVART

By way of application, imagine a wire running from left to right with a positive charge density of $+\frac{e}{\ell_o}$ and a negative charge density of $-\frac{e}{\ell_o}$, where $\ell_o$ is the distance between charges of given sign. On a chunk of wire of length $ds$, this translates to a positive charge $Q = \frac{e}{\ell_o}ds$

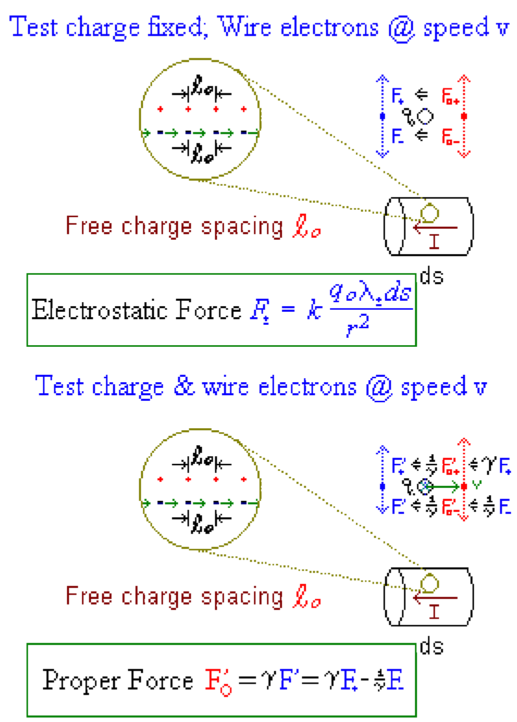


FIG. 5: Use of the frame-invariant proper-force in a map-based derivation of Biot-Savart.

and a negative charge of $-\frac{e}{\ell_o}ds$ and therefore a net charge of 0. If the positive charges are stationary in the wire, but the negative charges are moving to the right with a speed $v$, we say that the current in the wire is $I = \frac{e}{\ell_o}v$, to the left. From Coulomb's law, the electrostatic force on a stationary test charge $q$ a distance $r$ above the wire is of course

$$F_{up} = F_+ + F_- = k\frac{qQ}{r^2} - k\frac{qQ}{r^2} = 0 \qquad \text{(A1)}$$

Since the test-charge is not moving, the flat-space version of Newton's 2nd Law (here $F = F_o/\gamma$) predicts the same canceling proper-forces on our test charge as well. However if we now move the test charge to the right at a speed $v$, the 2nd Law predicts that the proper-force ($F_{o+}$) exerted on our test charge by the stationary *positive charges* remains in the direction of their separation, but *increases* in magnitude by a factor of $\gamma$. By symmetry, the co-moving frame-variant force due to negative charges (which now see the test charge as stopped) will have its previous contribution to the proper-force ($F_{o-}$) *decreased* by a factor of $1/\gamma$.

This proper-force provides a frame-invariant platform for combining the two frame-variant but otherwise purely *electrostatic* forces, allowing us (as illustrated in Fig. 5) to conclude that the net proper-force experienced by our moving test charge equals the proper-force from the positive charges on the test charge before it began moving ($kqQ/r^2$) times the non-zero difference between $\gamma$ and $1/\gamma$. Finally the net frame-variant force on our moving test particle, reduced from the net proper-force again by that factor of $\gamma$, becomes...

$$F_{up} = \frac{1}{\gamma}(\gamma - \frac{1}{\gamma})k\frac{qQ}{r^2} = k\frac{qQ}{r^2}\frac{v^2}{c^2} = qv\left(\frac{k}{c^2}\frac{Ids}{r^2}\right) \quad \text{(A2)}$$

This force, due to a frame-dependence of forces in spacetime that has nothing to do with electrostatic forces *per se*, is traditionally explained by saying that the current creates a magnetic field $B$ according to the Biot-Savart prescription in parentheses on the right, which in turn exerts a force according to the Lorentz Law $F = qvB$. Dan Schroeder in his talk on "Purcell simplified" at the 1999 winter AAPT meeting, followed upon long tradition and used multiple frames to go further, illustrating how magnetic fields are a convenient tool for tracking relativistic effects of the Coulomb force. We show here how the flat-space (one-map two-clock) version of Newton's 2nd law provides us with a derivation that (except for the symmetry invocation) requires only one map-frame with yardsticks and synchronized clocks. Moreover the illustration in Fig. 5, using frame-invariant proper-force along with its frame-variant cousin, compares favorably in its simplicity.