Bivariate Box Spline Wavelets in Sobolev Spaces

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ABSTRACT

We use bivariate boxspline functions to construct nonseparable wavelets in Sobolev spaces.

Keywords: Nonseparable, Orthonormal, Box Splines, Wavelet, Sobolev Spaces

1. INTRODUCTION

We are interested in constructing wavelets in Sobolev spaces in the bivariate setting. Our motivation and goal are to be able to numerically solve linear elliptic partial differential equations without inverting their linear systems when the equations are discretized using Galerkin's method. This paper is our initial step towards our goal. In this paper we first give some necessary and sufficient conditions for functions such that their integer translates form an orthonormal set in a Sobolev space. We shall also give conditions on subspaces to generate a multiresolution approximation (MRA) in a Sobolev space and show how to construct the wavelets associated with an MRA.

We use bivariate box spline functions to construct scaling functions satisfying the conditions for the orthonormality and generating a bonefide multiresolution approximation in a Sobolev space. We then construct their wavelets. Our construction generalizes the box spline wavelets in $L_2(\mathbf{R}^2)$ (cf. [1]) to Sobolev spaces and extends the Sobolev wavelets in the univariate setting (cf. [2]) to the bivariate setting.

2. PRELIMINARIES

Let s be an nonnegative real number and let

$$H^s(\mathbf{R}^2) = \{f : \|f\|_s < \infty\}$$

be the usual Sobolev space which is equipped with norm $\|\cdot\|_s$ defined by

$$\|f\|_s^2 = \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s|\hat{f}(\xi,\eta)|^2 d\xi d\eta.$$

Clearly, the inner product $\langle\cdot,\cdot\rangle_s$ associated with $\|\cdot\|_s$ is

$$\langle f,g\rangle_s = \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s \hat{f}(\xi,\eta)\overline{\hat{g}(\xi,\eta)}d\xi d\eta.$$

Proposition 2.1. *Let $\varphi \in H^s(\mathbf{R}^2)$. Then $\{2^j\varphi(2^jx-\ell_1, 2^jy-\ell_2),\ (\ell_1,\ell_2)\in Z^2\}$ is an orthonormal set in $H^s(\mathbf{R}^2)$ if and only if*

$$1 = \sum_{(\ell_1,\ell_2)\in Z^2}\left(1+2^{2j}(\xi+2\ell_1\pi)^2+2^{2j}(\eta+2\ell_2\pi)^2\right)^s |\hat{\varphi}(\xi+2\ell_1\pi,\eta+2\ell_2\pi)|^2 .$$

Part of the SPIE Conference on Wavelet Applications in Signal and Image Processing VI • San Diego, California • July 1998
SPIE Vol. 3458 • 0277-786X/98/$10.00

Proof: We have

$$\begin{aligned}
&\langle 2^j\varphi(2^jx-\ell_1,2^jy-\ell_2\rangle,2^j\varphi(2^jx,2^jy)\rangle_s\\
=\ &\frac{2^{2j}}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s|\hat\varphi(2^{-j}\xi,2^{-j}\eta)|^2e^{-i2^{-j}(\ell_1\xi+\ell_2\eta)}d\xi d\eta\\
=\ &\frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}\left(1+2^{2j}(\xi^2+\eta^2)\right)^s|\hat\varphi(\xi,\eta)|^2e^{-i(\xi\ell_1+\eta\ell_2)}d\xi d\eta\\
=\ &\frac{1}{(2\pi)^2}\int\int_{[0,2\pi]^2}\Bigg[\sum_{(n_1,n_2)\in Z^2}\left(1+2^{2j}(\xi+2n_1\pi)^2+2^{2j}(\eta+2n_2\pi)^2\right)^s\\
&\qquad\times|\hat\varphi(\xi+2n_1\pi,\eta+2n_2\pi)|^2\Bigg]e^{-i(\xi\ell_1+\eta\ell_2)}d\xi d\eta
\end{aligned}$$

Thus, $\langle 2^j\varphi(2^jx-\ell_1,2^jy-\ell_2\rangle=\delta_{\ell_1}\delta_{\ell_2},\ \forall(\ell_1,\ell_2)\in Z^2$ if and only if

$$\sum_{(n_1,n_2)\in Z^2}\left(1+2^{2j}(\xi+2n_1\pi)^2+2^{2j}(\eta+2n_2\pi)^2\right)^s|\hat\varphi(\xi+2n_1\pi,\eta+2n_2\pi)|^2=1$$

Proposition 2.2. *Let $\varphi_j,\ j\in Z$ be a sequence of functions in $H^s(\mathbf{R}^2)$ such that for each j, $\varphi_{j,(\ell_1,\ell_2)}=2^j\varphi_j(2^jx-\ell_1,2^jy-\ell_2)$, $(\ell_1,\ell_2)\in Z^2$ are orthornomal in $H^s(\mathbf{R}^2)$. Let*

$$V_j=\text{span}_{H^s(\mathbf{R}^2)}\{2^j\varphi_{j,(\ell_1,\ell_2)},(\ell_1,\ell_2)\in Z^2\}.$$

Suppose that $\lim_{j\to+\infty}|\hat\varphi_j(2^{-j}\xi,2^{-j}\eta)|=(1+\xi^2+\eta^2)^{-s/2}$. Then the union of the V_j's is dense in $H^s(\mathbf{R}^2)$.

Proof. Let P_j be the orthogonal projection from $H^s(\mathbf{R}^2)$ to V_j. That is, for $h\in H^s(\mathbf{R}^2)$

$$P_jh=\sum_{(\ell_1,\ell_2)\in Z^2}\langle h,\varphi_{j(\ell_1,\ell_2)}\rangle_s\varphi_{j,(\ell_1,\ell_2)}.$$

Then we have

$$\begin{aligned}
&\|P_jh\|_s^2=\sum_{(\ell_1,\ell_2)\in Z^2}|\langle h,\varphi_{j,(\ell_1,\ell_2)}\rangle_s|^2\\
=\ &\frac{2^{-2j}}{(4\pi^2)^2}\sum_{(\ell_1,\ell_2)\in Z^2}\left|\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s\hat h(\xi,\eta)\overline{\hat\varphi_j(2^{-j}\xi,2^{-j}\eta)}e^{-i2^{-j}(\ell_1\xi+\ell_2\eta)}d\xi d\eta\right|^2\\
=\ &\frac{2^{-2j}}{(2\pi)^4}\sum_{(\ell_1,\ell_2)\in Z^2}\Bigg|\sum_{(n_1,n_2)\in Z^2}\int\int_{[0,2^{j+1}\pi]^2}\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^s\\
&\times\hat h(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)\overline{\hat\varphi_j(2^{-j}\xi+2n_1\pi,2^{-j}\eta+2n_2\pi)}e^{-i2^{-j}(\ell_1\xi+\ell_2\eta)}d\xi d\eta\Bigg|^2\\
=\ &\frac{1}{(2\pi)^2}\int\int_{[0,2^{j+1}\pi]^2}\Bigg|\sum_{(n_1,n_2)\in\mathbf{Z}^2}\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^s\\
&\times\hat h(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)\overline{\hat\varphi_j(2^{-j}\xi+2n_1\pi,2^{-j}\eta+2n_2\pi)}\Bigg|^2d\xi d\eta
\end{aligned}$$

$$= \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}\sum_{(n_1,n_2)\in\mathbf{Z}^2}\left(1+\xi^2+\eta^2\right)^s(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\Big)^s\hat{h}(\xi,\eta)$$
$$\times\overline{\hat{h}(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)}\hat{\varphi}_j(2^{-j}\xi+2n_1\pi,2^{-j}\eta+2n_2\pi)d\xi d\eta$$

For $h\in C_0^\infty(\mathbf{R}^2)$, we know $|\hat{h}(\xi,\eta)|\le C(1+\xi^2+\eta^2)^{-\alpha}$ for any $\alpha>0$. By Proposition 2.1, $|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|\le C\ (1+\xi^2+\eta^2)^{-s/2}$. Thus, we have

$$\begin{aligned}
&\sum_{(n_1,n_2)\neq 0}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^s\Big|\hat{h}(\xi,\eta)\\
&\quad\times\overline{\hat{h}(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)}\cdot\overline{\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)}\hat{\varphi}_j(2^{-j}\xi+2n_1\pi,2^{-j}\eta+2n_2\pi)\Big|d\xi d\eta\\
\le&\sum_{(n_1,n_2)\neq 0}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{s/2}\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^{s/2}\\
&\quad\times\left|\hat{h}(\xi,\eta)\hat{h}(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)\right|d\xi d\eta\\
\le&\ C\sum_{(n_1,n_2)\neq 0}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{-4}\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^{-2}d\xi d\eta\\
\le&\ C\sum_{(n_1,n_2)\neq 0}\frac{1}{(1+(2^{j+1}n\pi)^2+(2^{j+1}n_2\pi)^2)^2}\int\int_{\mathbf{R}^2}\frac{1}{(1+\xi^2+\eta^2)^2}d\xi d\eta\\
\le&\ C\sum_{(n_1,n_2)\neq 0}\frac{1}{((2^{j+1}n_1\pi)^2+(2^{j+1}n_2\pi)^2)^2}=C2^{-4(j+1)}\sum_{(n_1,n_2)\neq 0}\frac{1}{(n_1^2+n_2^2)^2}\\
&\longrightarrow 0,\quad\text{as}\quad j\longrightarrow+\infty.
\end{aligned}$$

For $(n_1,n_2)=(0,0)$, we have

$$\begin{aligned}&\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{2s}|\hat{h}(\xi,\eta)|^2|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|^2d\xi d\eta\\
&\longrightarrow\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s|\hat{h}(\xi,\eta)|^2d\xi d\eta\end{aligned}$$

by Lebesgue's dominated convergence theorem. Hence, we have

$$||P_jh||_s^2\longrightarrow||h||_s^2\qquad\text{as}\quad j\longrightarrow+\infty$$

for $h\in C_0^\infty(\mathbf{R}^2)$. We now apply the following density argument to prove the above convergence holds for any $h\in H^s(\mathbf{R}^2)$.

We first recall the inequality

$$||f+g||_s^2\le(1+\delta)||f||_s^2+(1+\frac{1}{\delta})||g||_s^2.$$

for any norm or semi-norm and for any $\delta>0$. For any $\chi\in C_0^\infty(\mathbf{R}^2)$, we have

$$||P_jh||_s^2-\frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{2s}|\hat{h}(\xi,\eta)|^2|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|^2d\xi d\eta$$

$$\begin{aligned}
&\le (1+\delta)\|P_j\chi\|_s^2 + (1+\frac{1}{\delta})\|P_j(h-\chi)\|_s^2 \\
&-\frac{1}{(2\pi)^2}\frac{1}{(1+\delta)}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{2s}|\hat{\chi}(\xi,\eta)|^2|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|^2 d\xi d\eta \\
&+\frac{1}{(2\pi)^2}\frac{1}{\delta}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{2s}|\hat{h}(\xi,\eta)-\hat{\chi}(\xi,\eta)|^2|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|^2 d\xi d\eta \\
&\le (1+\delta)\left[\|P_j\chi\|_s^2 - \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{2s}|\hat{\chi}(\xi,\eta)|^2|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|^2 d\xi d\eta\right] \\
&+(1+\delta-\frac{1}{1+\delta})C\|\chi\|_s^2 + (1+\frac{1}{\delta})\|h-\chi\|_s^2 + \frac{C}{\delta}\|h-\chi\|_s^2.
\end{aligned}$$

For any $\varepsilon > 0$, we choose δ small enough such that $(1+\delta-\frac{1}{1+\delta})C\|\chi\|_s^2 < \varepsilon/2$ since we know an estimate of $\|\chi\|_s$. Then let χ be sufficiently close to h such that $(1+\frac{1}{\delta}+\frac{C}{\delta})\|h-\chi\|_s^2 < \varepsilon/4$. Finally, for χ, let j be sufficiently large,

$$(1+\delta)\left(\|P_j\chi\|_s^2 - \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{2s}|\hat{\chi}(\xi,\eta)|^2|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|^2 d\xi d\eta\right) < \varepsilon/4.$$

Similarly, we can obtain a lower bound.

That is,

$$\begin{aligned}
&\|P_jh\|_s^2 - \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{2s}|\hat{h}(\xi,\eta)|^2|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|^2 d\xi d\eta \\
&\longrightarrow 0 \quad \text{as} \quad j \longrightarrow +\infty
\end{aligned}$$

Since $|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)| \le C(1+\xi^2+\eta^2)^{-s/2}$, by Lebesgue's dominated convergence theorem, the second term above converges to $\|h\|_s^2$ for any $h \in H^s(\mathbf{R}^2)$. That is, $\|P_jh\|_s^2 \longrightarrow \|h\|_s^2$ as $j \to +\infty$. Therefore,

$$\begin{aligned}
\|P_jh - h\|_s^2 &= \|P_jh\|_s^2 - 2\langle P_jh, h\rangle_s + \|h\|_s^2 \\
&= \|h\|_s^2 - \|P_jh\|_s^2 \longrightarrow 0.
\end{aligned}$$

This completes the proof.

Proposition 2.3. *Let $\varphi_j, j \in Z$ be a sequence in $H^s(\mathbf{R}^2)$. Suppose that for each j, $\{2^j\varphi_j(2^jx-\ell_1, 2^jy-\ell_2), (\ell_1,\ell_2) \in Z^2\}$ forms an orthonormal basis for V_j. Suppose that there exists $\alpha > 0$ such that*

$$\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^{\alpha}|\hat{\varphi}_j(\xi,\eta)|^2 d\xi d\eta \le A$$

for every $j \le 0$, then $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$.

Proof. We may assume $j \le -1$. From the proof above, we have, by Proposition 2.1,

$$\|P_jh\|_s^2$$

$$
\begin{aligned}
= & \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}\sum_{(n_1,n_2)\in Z^2}(1+\xi^2+\eta^2)^s\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^s\hat{h}(\xi,\eta)\\
& \times\overline{\hat{h}(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)}\hat{\varphi}_j(2^{-j}\xi+2n_1\pi,2^{-j}\eta+2n_2\pi)d\xi d\eta\\
\leq & \frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s|\hat{h}(\xi,\eta)\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|\times\\
& \left(\sum_{(n_1,n_2)\in Z^2}\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^s\left|\hat{h}(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)\right|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

As $j\longrightarrow-\infty$, we know

$$
\begin{aligned}
& \sum_{n_1,n_2\in Z}\left(1+(\xi+2^{j+1}n_1\pi)^2+(\eta+2^{j+1}n_2\pi)^2\right)^s\left|\hat{h}(\xi+2^{j+1}n_1\pi,\eta+2^{j+1}n_2\pi)\right|^2 2^{2j+2}\pi^2\\
& \longrightarrow\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s|\hat{h}(\xi,\eta)|^2d\xi d\eta
\end{aligned}
$$

for $h\in C_0^\infty(\mathbf{R}^2)$. Thus, we have

$$
\begin{aligned}
& \|P_jh\|_s^2\leq\frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s|\hat{h}(\xi,\eta)\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)|C\|h\|_s^2 2^{-j}d\xi d\eta\\
\leq & \left(\frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}\left(1+2^{-2j}(|\xi|^2+|\eta|^2)\right)^\alpha\left|\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)\right|^2d\xi d\eta\right)^{1/2}2^{-j}C\|h\|_s^2\\
& \times\left(\frac{1}{(2\pi)^2}\int\int_{\mathbf{R}^2}\left(1+2^{-2j}(|\xi|^2+|\eta|^2)\right)^{-\alpha}(1+|\xi|^2+|\eta|^2)^{2s}|\hat{h}(\xi,\eta)|^2d\xi d\eta\right)^{1/2}\\
\leq & \frac{C\sqrt{A}\|h\|_s^2}{(2\pi)^2}\left(\int\int_{\mathbf{R}^2}\left(1+2^{-2j}(|\xi|^2+|\eta|^2)\right)^{-\alpha}(1+|\xi|^2+|\eta|^2)^{2s}|\hat{h}(\xi,\eta)|^2d\xi d\eta\right)^{1/2}\\
& \longrightarrow 0
\end{aligned}
$$

as $j\rightarrow-\infty$ for any $h\in C_0^\infty(\mathbf{R}^2)$. Thus, for $h\in H^s(\mathbf{R}^2)$, let χ be in $C_0^\infty(\mathbf{R}^2)$, we have

$$
\|P_jh\|_s\leq\|P_j(h-\chi)\|_s+\|P_j\chi\|_s\leq\|h-\chi\|_s+\|P_j\chi\|_s.
$$

We first choose χ very close to h and then let $j\rightarrow-\infty$. This completes the proof.

We now define a multiresolution approximation of $H^s(\mathbf{R}^2)$ as follows.

Definition 2.4. *Let $s>0$ be a real number. A multiresolution approximation (MRA) of $H^s(\mathbf{R}^2)$ is a sequence of subspaces V_j, $j\in Z$ such that*

1. $V_j\subset V_{j+1},\quad\forall j\in Z$;
2. $\bigcup_{j=-\infty} V_j$ *is dense in* $H^s(\mathbf{R}^2)$;
3. $\bigcap_{j=-\infty}^{+\infty} V_j=\{0\}$;

4. *for every $j \in Z$, there is a function $\varphi_j \in H^s(\mathbf{R}^2)$ such that the distributions $\varphi_j(\ell_1, \ell_2) = 2^j\varphi_j(2^jx - \ell_1, 2^jy - \ell_2)$, $(\ell_1, \ell_2) \in Z^2$ constitute an orthonormal basis for V_j.*

Furthermore, we call $\psi_j, j \in Z$ are wavelets in $H^s(\mathbf{R}^2)$ if $\psi_{j,(\ell_1,\ell_2)} = 2^j\psi_j(2^jx - \ell_1, 2^jy - \ell_2), j \in Z$, $(\ell_1, \ell_2) \in Z^2$ constitute an orthonormal basis for $H^s(\mathbf{R}^2)$.

To construct wavelets ψ_j $j \in Z$ in $H^s(\mathbf{R}^2)$, we start with an MRA. Suppose that $\varphi_j \in V_j$ such that $\varphi_{j,(\ell_1,\ell_2)} = 2^j\varphi_j(2^jx - \ell_1, 2^jy - \ell_2)$, $(\ell_1, \ell_2) \in Z^2$ form an orthonormal basis for V_j. Since $V_j \subset V_{j+1}$, we have

$$\hat{\varphi}_j(2\xi, 2\eta) = m_{j+1}(\xi, \eta)\hat{\varphi}_{j+1}(\xi, \eta)$$

for a 2π periodic function m_{j+1}.

Proposition 2.5. *Let m_j be defined above. Then*

$$\sum_{\ell=0}^{3} |m_j((\xi, \eta) + \pi_\ell)|^2 = 1 \qquad \forall(\xi, \eta) \in \mathbf{R}^2$$

where $\pi_0 = (0, 0)$, $\pi_1 = (\pi, 0)$, $\pi_2 = (0, \pi)$ and $\pi_3 = (\pi, \pi)$.

Proof. By Proposition 2.1, we have

$$\begin{aligned}
1 &= \sum_{\ell_1,\ell_2)\in Z^2} \left(1 + 2^{2j}(2\xi + 2\pi\ell_1)^2 + 2^{2j}(2\eta + 2\pi\ell_2)^2\right)^s |\hat{\varphi}_j(2\xi + 2\pi\ell_1, 2\eta + 2\pi\ell_2)|^2 \\
&= \sum_{(\ell_1,\ell_2)\in Z^2} \left(1 + 2^{2j+2}(\eta + \pi\ell_1)^2 + 2^{2j+2}(\eta + \pi\ell_2)^2\right)^s \\
&\qquad \times |m_j(\xi + \pi\ell_1, \pi + \pi\ell_2)|^2 \, |\hat{\varphi}_j(\xi + \pi\ell_1, \eta + \pi\ell_2)|^2 \\
&= \sum_{\ell=0}^{3} |m_j((\xi, \eta) + \pi_\ell)|^2
\end{aligned}$$

which completes the proof.

Suppose that we have m_j, $j \in Z$ satisfying the condition in Proposition 2.5 which is a necessary condition for the orthonormality. We may define $\hat{\varphi}_j$ by

$$\begin{aligned}
\hat{\varphi}_j(\xi, \eta) &= m_{j+1}(\xi/2, \eta/2)\hat{\varphi}_{j+1}(\xi/2, \eta/2) \\
&= \prod_{p=1}^{J} m_{j+p}(\xi/2^p, \eta/2^p)\hat{\varphi}_{j+J}(\xi/2^J, \eta/2^J) \\
&= \cdots = \frac{1}{(1 + \xi^2 + \eta^2)^{s/2}} \prod_{p=1}^{+\infty} m_{j+p}(\xi/2^p, \eta/2^p) \qquad (2.1)
\end{aligned}$$

for $j \in Z$. For V_j, let W_j be the orthogonal complement of V_j in V_{j+1}. For m_j, let $n_{j,1}$, $n_{j,2}$, $n_{j,3}$ be three $(2\pi)^2$ periodic functions such that

$$\begin{bmatrix}
m_j(\xi, \eta) & m_j(\xi + \pi, \eta) & m_j(\xi, \eta + \pi) & m_j(\xi + \pi, \eta + \pi) \\
n_{j,1}(\xi, \eta) & n_{j,1}(\xi + \pi, \eta) & n_{j,1}(\xi, \eta + \pi) & n_{j,1}(\xi + \pi, \eta + \pi) \\
n_{j,2}(\xi, \eta) & n_{j,2}(\xi + \pi, \eta) & n_{j,2}(\xi, \eta + \pi) & n_{j,2}(\xi + \pi, \eta + \pi) \\
n_{j,3}(\xi, \eta) & n_{j,3}(\xi + \pi, \eta) & n_{j,3}(\xi, \eta + \pi) & n_{j,3}(\xi + \pi, \eta + \pi)
\end{bmatrix} \qquad (2.2)$$

is a unitary matrix. Then let

$$\hat{\psi}_{j,\ell}(2\xi,2\eta) = n_{j,\ell}(\xi,\eta)\hat{\varphi}_j(\xi,\eta), \quad \ell = 1,2,3. \tag{2.3}$$

for $j \in Z$. We have

Theorem 2.6. *Suppose that φ_j, $j \in Z$ defined by (2.1) generate an MRA $\{V_j\}$ of $H^s(\mathbf{R}^2)$ and $\varphi_{j,(\ell_1,\ell_2)}$, $(\ell_1,\ell_2) \in Z^2$ form an orthonormal basis for V_j, $j \in Z$. Suppose that for each $j \in Z$, m_j, $n_{j,1}$, $n_{j,2}$, $n_{j,3}$ such that the matrix (2.2) is unitary. Defined $\psi_{j,\ell}$ by (2.3) for $\ell = 1,2,3$ and $j \in Z$. Then $W_j = W_{j,1} \oplus W_{j,2} \oplus W_{j,3}$ with $W_{j,\ell} = \mathrm{span}_{H^s(\mathbf{R}^2)}\{2^j\psi_{j,\ell}(2^jx - i_1, 2^jy - i_2), (i_1,i_2) \in Z^2\}$, $\ell = 1,2,3$ is perpendicular to V_j in V_{j+1} and $V_{j+1} = V_j \oplus W_j$. Therefore,*

$$\{2^j\psi_{j,\ell}(2^jx - i_1, 2^jy - i_2), (i_1,i_2) \in Z^2, j \in Z, \ell = 1,2,3\}$$

is an orthonormal basis for $H^s(\mathbf{R}^2)$.

Proof. We first prove that $2^j\psi_{j,\ell}(2^jx - i_1, 2^jy - i_2) \perp V_j$ for all $(i_1,i_2) \in Z$ and $\ell = 1,2,3$. Indeed,

$$\begin{aligned}
& (2\pi)^2\langle 2^j\psi_{j,\ell}(2^jx - i_1, 2^jy - i_2), 2^j\varphi_j(2^jx - \ell_1, 2^jy - \ell_2)\rangle_s \\
= \; & 2^{-2j}\int\int_{\mathbf{R}^2}(1+\xi^2+\eta^2)^s\hat{\psi}_{j,\ell}(2^{-j}\xi, 2^{-j}\eta)\overline{\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta)}e^{-i2^{-j}(\xi(i_1-\ell_1)+\eta(i_2-\ell_2))}d\xi d\eta \\
= \; & \int\int_{\mathbf{R}^2}(1+2^{2j}(\xi^2+\eta^2))^s\hat{\psi}_{j,\ell}(\xi,\eta)\overline{\hat{\varphi}_j(\xi,\eta)}e^{-i(\xi(i_1-\ell_1)+\eta((i_2-\ell_2))}d\xi d\eta \\
= \; & \int\int_{[0,2\pi]^2}\sum_{(p_1,p_2)\in Z^2}\left(1+2^{2j}((\xi+2\pi p_1)^2+(\eta+2\pi p_2)^2)\right)^s n_{j,\ell}(\frac{\xi}{2}+\pi p_1, \frac{\eta}{2}+\pi p_2) \\
& \times\overline{m_j(\xi/2+\pi p_1, \eta/2+\pi p_2)|\hat{\varphi}_{j+1}(\xi/2+\pi p_1, \eta/2+\pi p_2))|^2}e^{-i(\xi(i_1-\ell_1)+\eta(i_2-\ell_2))}d\xi d\eta \\
= \; & \int\int_{[0,2\pi]^2}\left[\sum_{p=0}^{3} n_{j,\ell}((\xi/2,\eta/2)+\pi_p)\overline{m_j((\xi/2,\eta/2)+\pi_p)}\right]e^{-i\xi(i_1-\ell_1)+\eta(i_2-\ell_2))}d\xi d\eta
\end{aligned}$$

by Proposition 2.1. Since the matrix (2.2) is unitary, we get that the integrand above is zero and then $W_{j,\ell} \perp V_j$. Similar to the argument above, we can see that $\{2^j\psi_{j\ell}(2^jx - i_1, 2^jy - i_2), (i, i_2) \in Z^2\}$ is an orthonormal basis for $W_{j,\ell}$, $\ell = 1,2,3$.

We now show that $V_{j+1} = V_j \oplus W_{j,1} \oplus W_{j,2} \oplus W_{j,3}$. For any $f \in V_{j+n}$, we write

$$\hat{f}(\xi,\eta) = C(2^{-j-1}\xi, 2^{-j-1}\eta)\hat{\varphi}_{j+1}(2^{-j-1}\xi, 2^{-j-1}\eta).$$

We shall demonstrate that there exist $A(2^{-j}\xi, 2^{-j}\eta)$ and $B_\ell(2^{-j}\xi, 2^{-j}\eta)$ which are $2\pi \times 2\pi$ periodic functions such that

$$\hat{f}(\xi,\eta) = A(2^{-j}\xi,2^{-j}\eta)\hat{\varphi}_j(2^{-j}\xi,2^{-j}\eta) + \sum_{\ell=1}^{3} B_\ell(2^{-j}\xi,2^{-j}\eta)\hat{\psi}_{j,\ell}(2^{-j}\xi,2^{-j}\eta).$$

That is, we have

$$\begin{aligned}
& C(\xi/2,\eta/2)\hat{\varphi}_{j+1}(\xi/2,\eta/2) \\
= \; & A(\xi,\eta)\hat{\varphi}_j(\xi,\eta) + \sum_{\ell=1}^{3} B_\ell(\xi,\eta)\hat{\psi}_{j,\ell}(\xi,\eta)
\end{aligned}$$

$$= \quad A(\xi,\eta)m_{j+1}(\xi/2,\eta/2)\hat{\psi}_{j+1}(\xi/2,\eta/2) + \sum_{\ell=1}^{3} B_\ell(\xi,\eta)n_{j+1,\ell}(\xi/2,\eta/2)\hat{\varphi}_{j+1}(\xi/2,\eta/2).$$

It follows that

$$C(\xi/2,\eta/2) = A(\xi,\eta)m_{j+1}(\xi/2,\eta/2) + \sum_{\ell=1}^{3} B_\ell(\xi,\eta)n_{j+1,\ell}(\xi/2,\eta/2).$$

Note that A and B_ℓ's are $2\pi \times 2\pi$ periodic functions, we have

$$C((\xi/2,\eta/2)+\pi_p) = A(\xi,\eta)m_{j+1}((\xi/2,\eta/2)+\pi_p) + \sum_{\ell=1}^{3} B_\ell(\xi,\eta)n_{j+1,\ell}((\xi/2,\eta/2)+\pi_p)$$

for $p = 0,1,2,3$. Since the matrix (2.2) is unitary, we know that $A(\xi,\eta)$, $B_\ell(\xi,\eta)$, $\ell = 1,2,3$, have a unique solution. This completes the proof.

3. CONSTRUCTION OF BOXSPLINE WAVELETS IN SOBOLEV SPACES

With the background knowledge on wavelets in Sobolev spaces, let us construct them using bivariate boxspline functions. Fix $s \geq 0$ and natural numbers (l,m,n) such that

$$\min\{\ell+m-2, \ell+n-2, m+n-2\} > s.$$

Let $B_{\ell,m,n}$ be a box spline function defined in terms of Fourier transform by

$$\hat{B}_{\ell,m,n}(\xi,\eta) = e^{i\xi(\ell+n)/2}e^{i\eta(m+n)/2}\left(\frac{\sin\xi/2}{\xi/2}\right)^\ell \left(\frac{\sin\eta/2}{\eta/2}\right)^m \left(\frac{\sin\left(\frac{\xi+\eta}{2}\right)}{(\xi+\eta)/2}\right)^n$$

Then, $B_{\ell,m,n} \in H^s(\mathbf{R}^2)$ (cf. [3] and [4]). Let

$$\omega_{\ell,m,n,j}(\xi,\eta) = \sum_{(\ell_1,\ell_2)\in Z^2} (1+2^{2j}((\xi+2\pi\ell_1)^2+(\eta+2\pi\ell_2)^2))^s |\hat{B}_{\ell,m,n}(\xi+2\pi\ell_1, \eta+2\pi\ell_2)|^2.$$

It is known that there exist two constants $\underline{k}_{\ell,m,n}$ and $\overline{k}_{\ell,m,n}$ such that

$$0 \leq \underline{k}_{\ell,m,n} \leq \sum_{(\ell_1,\ell_2)\in Z^2} |\hat{B}_{\ell,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \leq \overline{k}_{\ell,m,n} < +\infty$$

(cf. [5].)

It is easy to see $\omega_{\ell,m,n,j}(\xi,\eta) \geq \underline{k}_{\ell,m,n}$ and

$$\begin{aligned}\omega_{\ell,m,n,j}(\xi,\eta) \quad \leq \quad & 3^s \Bigg(\overline{k}_{\ell,m,n} + 2^{2js} \sum_{(\ell_1,\ell_2)\in Z^2} (\xi+2\pi\ell_1)^{2s} |\hat{B}_{\ell,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \\ & + 2^{2js} \sum_{(\ell_1,\ell_2)\in Z^2} (\eta+2\pi\ell_2)^{2s} |\hat{B}_{\ell,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \Bigg).\end{aligned}$$

We need the following

Lemma 3.1. *We have for any* $(x,y) \in \mathbf{R}^2$,

$$\frac{1+x^2}{(1+y^2)(1+(x+y)^2)} \leq 4 \text{ or } \frac{x^2}{(1+y^2)(1+(x+y)^2)} \leq 1.$$

Since $\omega_{\ell,m,n,j}(\xi,\eta)$ is a $2\pi \times 2\pi$ periodic function, we consider $(\xi,\eta) \in [0,2\pi]^2$ and ℓ_1, ℓ_2 are sufficiently large. Then, we use Lemma 3.1 to have, for $|\xi + 2\pi\ell_1 + \eta + 2\pi\ell_2| \geq 1$,

$$\begin{aligned} (\xi+2\pi\ell_1)^{2s} \quad & |\hat{B}_{\ell,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \\ \leq \quad & C(\xi+2\pi\ell_1)^{2(s-\ell)}\left(\frac{1}{1+(\eta+2\pi\ell_2)^2}\right)^m \left(\frac{1}{1+(\xi+2\pi\ell_1+\eta+2\pi\ell_2)^2}\right)^n \\ \leq \quad & C\left(\frac{1}{1+(\eta+2\pi\ell_2)^2)}\right)^{m-s+\ell} \left(\frac{1}{1+(\xi+2\pi\ell_1+\eta+2\pi\ell_2)^2}\right)^{n-s+\ell}. \end{aligned}$$

For $|\xi + 2\pi\ell_1 + \eta + 2\pi\ell_2| < 1$, we have $|\xi+2\pi\ell_1| \leq 1 + |\eta + 2\pi\ell_2|$ and

$$\begin{aligned} & (\xi+2\pi\ell_1)^{2s}|\hat{B}_{\ell,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \\ \leq \quad & C(\xi+2\pi\ell_1)^{2(s-\ell)}|\hat{B}_{0,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \\ \leq \quad & C|\hat{B}_{0,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 + |\hat{B}_{0,m+\ell-s,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \end{aligned}$$

These imply that

$$\sum_{(\ell_1,\ell_2)\in Z^2} (\xi+2\pi\ell_1)^{2s}|\widehat{B}_{\ell,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \leq C < +\infty$$

Similarly, we have

$$\sum_{(\ell_1,\ell_2)\in Z^2} (\eta+2\pi\ell_2)^{2s}|\widehat{B}_{\ell,m,n}(\xi+2\pi\ell_1,\eta+2\pi\ell_2)|^2 \leq C < +\infty$$

Hence we have

Lemma 3.2. *There exists two cosntants* $\underline{K}_{\ell,m,n}$ *and* $\overline{K}_{\ell,m,n}$ *such that*

$$0 < \underline{K}_{\ell,m,n} \leq \omega_{\ell,m,n,j}(\xi,\eta) \leq 2^{2j}\overline{K}_{\ell,m,n} < +\infty.$$

We now define for every $j \in Z$,

$$\hat{\varphi}_j(\xi,\eta) = \frac{\widehat{B}_{\ell,m,n}(\xi,\eta)}{\sqrt{\omega_{\ell,m,n,j}(\xi,\eta)}} \tag{3.1}$$

Then it is easy to see $\varphi_j \in H^s(\mathbf{R}^2)$ by Lemma 3.2. Moreover, these functions φ_j satisfy the condition in Propositions 2.1.

Next, we study $\hat{\varphi}_j(2^{-j}\xi, 2^{-j}\eta)$ for $j > 0$. Considering the terms in

$$\omega_{\ell,m,n,j}(2^{-j}\xi,2^{-j}\eta) \quad = \quad \sum_{\ell_1,\ell_2)\in Z^2} (1+(\xi+2^{j+1}\pi\ell_1)^2+(\eta+2^{j+1}\pi\ell_2)^2)^s$$

$$\times|\widehat{B}_{\ell,m,n}(\xi/2^j + 2\pi\ell_1, \eta/2^j + 2\pi\ell_2)|^2,$$

we have, for $(\ell_1, \ell_2) \neq (0,0)$, $(\xi, \eta) \in [0, 2\pi]^2$ and j large enough,

$$\begin{aligned} & \left(1 + (\xi + 2^{j+1}\pi\ell_1)^2 + (\eta + 2^{j+1}\pi\ell_2)^2\right)^s \left|\widehat{B}_{\ell,m,n}(\xi/2^j + 2\pi\ell_1, \eta/2^j + 2\pi\ell_2)\right|^2 \\ \leq\ & \left(1 + (\xi + 2^j\pi\ell_1)^2 + (\eta + 2^{j+1}\pi\ell_2)^2\right)^s \times \\ & \begin{cases} 2^{-2(j+1)(\ell+m+n)}|\xi|^{2\ell}|\eta|^{2m}|\xi+\eta|^{2n}, & \ell_1 \neq 0, \ell_2 \neq 0, \ell_1 + \ell_2 \neq 0, \\ 2^{-2(j+1)(\ell+n)}|\xi|^{2\ell}|\xi+\eta|^{2n}, & \ell_1 \neq 0, \ell_2 \neq 0, \\ 2^{-2(j+1)(m+n)}, & \ell_1 = 0, \ell_2 \neq 0, \\ 2^{-2(j+1)(\ell+n)}, & \ell_1 + \ell_2 = 0. \end{cases} \end{aligned}$$

The condition $min\{\ell+n-2, \ell+m-2, m+n-2\} > s$ implies each term in $\omega_{\ell,m,n,j}(\xi,\eta)$ converges to zero except for $(\ell_1, \ell_2) = (0,0)$. Thus,

$$\omega_{\ell,m,n,j}(2^{-j}\xi, 2^{-j}\eta) \longrightarrow (1 + \xi^2 + \eta^2)^s$$

as $j \to +\infty$. Since $|\widehat{B}_{\ell,m,n}(2^{-j}\xi, 2^{-j}\eta)| \to 1$. Hence, we have

$$\lim_{j\to+\infty} |\hat{\varphi}_j(2^{-j}\xi, 2^{-j}\eta)| = (1 + \xi^2 + \eta^2)^{-s/2}$$

That is, φ_j $j \in Z_+$ satisfy the conditions in Proposition 2.2.

Furthermore, we consider $\alpha = 1$, for $j < 0$

$$\begin{aligned} & \int\int_{\mathbf{R}^2} (1 + \xi^2 + \eta^2)^\alpha |\hat{\varphi}_j(\xi,\eta)|^2 d\xi d\eta \\ \leq\ & \frac{1}{\sqrt{\underline{K}_{\ell,m,n}}} \int\int_{\mathbf{R}^2} (1 + \xi^2 + \eta^2)^\alpha |\widehat{B}_{\ell,m,n}(\xi,\eta)|^2 d\xi d\eta \\ \leq\ & C\int\int_{\mathbf{R}^2} |\widehat{B}_{\ell,m,n}(\xi,\eta)|^2 d\xi d\eta + C\int\int_{\mathbf{R}^2} |\widehat{B}_{\ell-1,m,n}(\xi,\eta)|^2 d\xi d\eta \\ & +C\int\int_{\mathbf{R}^2} |\widehat{B}_{\ell,m-1,n}(\xi,\eta)|^2 d\xi d\eta < +\infty. \end{aligned}$$

Here, we have used Lemma 3.2. Hence, the conditions in Proposition 2.3 is satisfied.

Let $V_j = \text{span}_{H^s(\mathbf{R}^2)}\{\varphi_j(2^j x - \ell_1, 2^j y - \ell_2), (\ell_1, \ell_2) \in Z^2\}$ for every $j \in Z$. Then the above arguments and discussion show

Theorem 3.3. *Let φ_j be defined in (3.1) for every $j \in Z$ and V_j be defined above. Then $\{V_j\}_{j\in Z}$ constitutes a multiresolution approximation of $H^s(\mathbf{R}^2)$.*

Finally, let us construct wavelets associated with the scaling functions $\varphi_j, j \in Z$. We define $2\pi \times 2\pi$ periodic functions $n_{j,1}$, $n_{j,2}$ and $n_{j,3}$ by

$$\begin{aligned} n_{j,1}(\xi,\eta) &= e^{i(\xi+\eta)} \begin{cases} m_j(\xi+\pi,\eta), & \text{if } (\ell+n) \text{ is even}, \\ \overline{m_j(\xi+\pi,\eta)}, & \text{if } (\ell+n) \text{ is odd}, \end{cases} \\ n_{j,2}(\xi,\eta) &= e^{i\eta} \begin{cases} m_j(\xi,\eta+\pi), & \text{if } (m+n) \text{ is even}, \\ \overline{m_j(\xi,\eta+\pi)}, & \text{if } (m+n) \text{ is odd}, \end{cases} \end{aligned}$$

$$n_{j,3}(\xi,\eta) = e^{i\xi}\begin{cases} m_j(\xi+\pi,\eta+\pi) & \text{if } (\ell+m) \text{ is even,} \\ \overline{m_j(\xi+\pi,\eta+\pi)} & \text{if } \ell+m \text{ is odd.} \end{cases}$$

We can verify that the following matrices

$$\begin{bmatrix} m_j(\xi,\eta) & m_j(\xi+\pi,\eta) & m_j(\xi,\eta+\pi) & m_j(\xi+\pi,\eta+\pi) \\ n_{j,1}(\xi,\eta) & n_{j,1}(\xi+\pi,\eta) & n_{j,1}(\xi,\eta+\pi) & n_{j,1}(\xi+\pi,\eta+\pi) \\ n_{j,2}(\xi,\eta) & n_{j,2}(\xi+\pi,\eta) & n_{j,2}(\xi,\eta+\pi) & n_{j,2}(\xi+\pi,\eta+\pi) \\ n_{j,3}(\xi,\eta) & n_{j,3}(\xi+\pi,\eta) & n_{j,3}(\xi,\eta+\pi) & n_{j,3}(\xi+\pi,\eta+\pi) \end{bmatrix}$$

are unitary for all $j \in Z$. The verification is the same as in [1]. We may omit the detail here.

We now construct nonseperable bivariate wavelets in $H^s(\mathbf{R}^2)$ by

$$\hat{\psi}_{j,k}(\xi,\eta) = n_{j,k}(\xi/2,\eta/2)\hat{\varphi}_j(\xi/2,\eta/2), k = 1,2,3.$$

When $s = 0$, these wavelets are exactly the wavelets constructed in [1].

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