2. Consider a particle of mass \( m \) confined to a circular box of radius \( a \). Show that the Schrödinger equation is separable in cylindrical coordinates \( \rho, \phi \) and find and solve the equations for the radial and the angular motion.

The Schrödinger equation inside the "box", i.e. \( \rho < a \) is:

\[
\frac{-\hbar^2}{2m} \left[ \frac{\partial^2}{\partial \rho^2} \frac{1}{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] \psi(\rho, \phi) = E \psi(\rho, \phi)
\]

Try letting \( \psi(\rho, \phi) = R(\rho) \Phi(\phi) \) see if this equation is separable, i.e.

\[
\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -\frac{m^2}{\hbar^2} \Phi
\]

We get 2 equations

(1) \( \frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi \) where \(-m^2\) is the separation constant. The solutions are \( \Phi(\phi) = e^{im\phi} \) where \( m = 0, \pm 1, \pm 2, \ldots \) (Thomson)

Making the substitution \( \rho = \sqrt{\frac{2mE}{\hbar^2}} \) (Thomson) = \( \sqrt{\frac{2mE}{\hbar^2}} \) (symbol)

the 2nd equation becomes

(2) \( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \frac{1}{r^2} (1 - \frac{m^2}{1}) R = 0 \) which is the Bessel equation of order \( m \).