c. Spin-Orbit Correction

The term "spin-orbit" refers to the interaction of the electron, considered to be a bar magnet, and the magnetic field experienced by the electron in its own rest frame. The spin-orbit correction is therefore nothing more than the energy of a magnetic dipole immersed in a magnetic field. This energy is given by

\[ E = -\mu_S \mathbf{\cdot B} \]

where \( \mu_S \) is the spin magnetic moment and \( B \) the field due to the proton motion about the electron. We already know \( \mu_S \) which is given by

\[ \text{Spin magnetic moment} = \mu_S = -\frac{g \mu_B S}{\hbar} \]

where \( \mu_B \) is the Bohr magneton given by

\[ \mu_B = \text{Bohr magneton} = \frac{e\hbar}{2mc} \quad (\text{Gaussian units}) \]

We need only evaluate the magnetic induction \( B \) as "seen" by the electron. Note that the energy, \(-\mu_S \mathbf{\cdot B}\), will be an operator and, because it is a fine-structure correction, will be of the same order of magnitude as the relativistic correction.

Taking the electronic velocity to be \( \mathbf{v} \), the velocity of the proton in the rest frame of the electron is \(-\mathbf{v}\). Then, according to the law of Biot and Savard, the magnetic induction due to the motion of the proton is

\[ B = \frac{(-\mathbf{ve}) \times \mathbf{r}}{r^3 c^2} \]

where \( \mathbf{r} \) is the vector from the electron to the proton. Now, we may cast \( B \) in terms of the electronic orbital angular momentum \( \mathbf{L} \) because \( \mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times mv \).

Note that we will then have \( B \) in terms of the orbital angular momentum and \( \mu \) in terms of the spin of the electron so the designation "spin-orbit interaction" is quite appropriate. The magnetic induction becomes
\[ B = \frac{eL}{mr^3c^2} \]

so that the energy, the correction term in the hamiltonian is

\[ E = \frac{e^2}{m^2r^3c^2}(S \cdot L) \]

This expression for the energy is almost correct. It is incorrect because the energy splitting seen in the laboratory frame is not the same as the splitting seen in the electron rest frame, a consequence of the fact that the electron is accelerating around the proton. The effect is known as Thomas precession. Happily, it changes our last result by a factor of 1/2 so the spin-orbit correction hamiltonian \( H_{SO} \) is

\[ H_{SO} = \left( \frac{1}{2} \right) \cdot \frac{e^2}{m^2r^3c^2}(S \cdot L) \]