a. Introduction

The degeneracies of the hydrogen have been attributed to the existence of symmetries inherent in the Coulomb potential. These symmetries must manifest themselves in a classical formulation of the problem so we examine the hydrogen atom from a classical point of view.

The Coulomb potential and the Newtonian gravitational potential are both proportional to 1/r. Therefore, the classical description of the hydrogen atom is directly comparable with that of the Kepler problem. From any classical mechanics book we find the equation of the orbit in the Kepler problem with potential \( V(r) = -k/r \). Since angular momentum is conserved for a central potential the motion is confined to a plane and we use plane polar coordinates \( r \) and \( \phi \). We have

\[
\frac{\alpha}{r} = 1 + \varepsilon \cos \phi
\]

where \( \varepsilon \) is the eccentricity. The \( \alpha \) that appears in this equation is not the fine-structure constant, but since this designation is universal, we retain this symbol. \( \alpha \) is, in fact, called the latus rectum and is given by

\[
\alpha = \frac{\ell^2}{\mu k}
\]

where \( \ell \) is the angular momentum and \( \mu \) the reduced mass of the system. \( \varepsilon \) is given by

\[
\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}
\]

where \( E \) is the total energy of the system. The nature of the trajectory of the particle of mass \( \mu \) depends upon the value of \( \varepsilon \) in the following way.
Clearly the only way the orbit can represent a bound state is if $E < 0$, thus making $\varepsilon$ less than unity.

The various quantities associated with the elliptical orbit are shown in the figure below. The sun (proton) is located at the focus $P$ and the earth (electron) executes the elliptical orbit. $r_{\text{min}}$ and $r_{\text{max}}$ are, respectively, the pericenter and apocenter of the orbit.

Now, using the figure we may find an expression for $a$ the semi-major and $b$ semi-minor axes.

$$a = \frac{\alpha}{1 - \varepsilon^2} = \frac{k}{2 |E|}$$

$$b = \frac{\alpha}{\sqrt{1 - \varepsilon^2}} = \frac{\ell}{\sqrt{2\mu |E|}}$$

The first of these equations is very important because it shows that the total energy depends only upon $a$ and not upon $b$.

$$E = -\frac{k}{2a}$$
where the absolute value has been replaced by a minus sign since we are interested only in the bound states. The semi-minor axis, $b$, is seen to depend upon both the energy and the angular momentum $\ell$. This shows that the energy does not depend upon the angular momentum, reminiscent of the accidental degeneracy we encountered in the quantum mechanical treatment. In fact, this is a case of a classical degeneracy, the energy is the same no matter what the angular momentum. There are an infinite number of orbits having the same energy, but semi-minor axes and hence different angular momenta. Obviously the classical and quantum mechanical degeneracies are related.