c. Solution of the separated equations - the energy eigenfunctions

In order to put the $\xi$- and $\eta$-equations in the same form we changed the variable to $\zeta = \alpha \xi$ and $\zeta = \alpha \eta$. This led to

$$f(\zeta) = \zeta^{(1/2)|m|} \exp\left[-\frac{1}{2} \zeta\right] L(\zeta)$$

and the equation

$$\zeta L''(\zeta) + (|m| + 1 - \zeta) L'(\zeta) + \left[\lambda - \frac{1}{2} (|m| + 1)\right] L(\zeta) = 0$$

The solutions of this equation for which the series is forced to terminate are the associated Laguerre polynomials, $L_{n+|m|}^{\lambda} (\zeta)$, with an analogous solution for the $\eta$-equation. Thus, the (unnormalized) wave function $\psi(\xi, \eta, \phi)$ is given by

$$\psi_{n_1, n_2, m}(\xi, \eta, \phi) = \exp\left[-\frac{\alpha(\xi + \eta)}{2}\right] (\xi \eta)^{|m|/2} \cdot L_{n_1+|m|}^{\lambda}(\xi \xi) \cdot L_{n_2+|m|}^{\lambda}(\eta \eta) \cdot \exp(\pm im\phi)$$

To conform to the wave function given in Bethe and Salpeter, we make the substitution $E = -\left(1/2\right)\epsilon^2$ and normalize. We obtain

$$\psi_{n_1, n_2, m}(\xi, \eta, \phi) = \frac{1}{\pi \Gamma(n_1 + m + 3/2)} \cdot \left\{ \frac{n_1! n_2! \epsilon^{m + 3/2}}{\Gamma(n_1 + m)! \Gamma(n_2 + m)!} \right\} \times \exp\left[-\frac{\alpha(\xi + \eta)}{2}\right] (\xi \eta)^{|m|/2} \cdot L_{n_1+|m|}^{\lambda}(\xi \xi) \cdot L_{n_2+|m|}^{\lambda}(\eta \eta) \cdot \exp(\pm im\phi)$$

Upon taking the absolute square of $\psi$, the $\phi$-dependence disappears just as it did in spherical coordinates. Therefore, the probability density is cylindrically symmetric about the z-axis.

As was done for the orbital wave functions, we may make a density plot of $|\psi|^2$. 

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The contrast between the charge density for a parabolic eigenstate, sometimes referred to as Stark eigenstates, and the orbital eigenstates is striking. While the charge densities for the orbital eigenstates must be symmetric about the origin, no such symmetry exists for the Stark eigenstates. This feature will be discussed in detail later in this text.