b. Solution of the separated equations - the energy eigenvalues

We may cast these equations in a more convenient form. In the $\xi$-equation let

$$\xi = \alpha \zeta$$

to obtain

$$\frac{1}{\zeta} \frac{d}{d\zeta} \left( \zeta \frac{df(\zeta)}{d\zeta} \right) + \left( \frac{\lambda_1}{\zeta} - \frac{1}{4} - \frac{m^2}{4\zeta^2} \right) f(\zeta) = 0$$

where

$$\lambda_1 = \frac{1}{\alpha} (1 - \nu) \quad \text{and} \quad \alpha^2 = 2|\lambda|$$

Notice that this equation contains the $\phi$ quantum number $m$ and that $\lambda_1$ contains the separation constant $\nu$. The energy is contained in both $\lambda_1$ and $\alpha$.

The $\eta$-equation can be similarly transformed. It has exactly the same form as the $\zeta$-equation above with $\lambda_1 \to \lambda_2$ and

$$\zeta = \alpha \eta \quad \text{and} \quad \lambda_2 = \frac{\nu}{\alpha}$$

We use the same technique to solve the $\zeta$-equation as that employed to solve the radial equation in spherical coordinates. Asymptotically

$$f(\zeta) \to \exp \left[ \pm \frac{i}{2} \zeta \right]$$

As usual, we must discard the plus sign. We try a solution of the form

$$F(\zeta) = \zeta^s \left( a_0 + a_1 \zeta^1 + a_2 \zeta^2 + \ldots \right) = \zeta^s L(\zeta) \exp \left[ -\frac{i}{2} \zeta \right]$$

and find that

$$s = \pm \left( \frac{1}{2} \right) m$$
but, since \( m \) may be positive or negative, we must require that

\[ s = -\left( \frac{1}{2} \right) |m| \]

and

\[ f(\zeta) = \zeta^{(1/2)|m|} \exp\left[ -\frac{1}{2} \zeta \right] L(\zeta) \]

which gives

\[ \zeta L''(\zeta) + (|m| + 1 - \zeta) L'(\zeta) + \left[ \lambda_1 - \frac{1}{2} (|m| + 1) \right] L(\zeta) = 0 \]

As in the solution to the radial equation in spherical coordinates, we find that the series must be terminated in order to keep the wave function from blowing up. It follows then that the quantity

\[ \lambda_1 - \frac{1}{2} (|m| + 1) \]

must be an integer or zero. Denoting this integer by \( n_1 \), we have

\[ n_1 = 0, 1, 2, ... = \lambda_1 - \frac{1}{2} (|m| + 1) \]

Now, the \( \eta \)-equation is virtually identical to the \( \xi \)-equation so that we already know the solution. Letting \( n_1 \rightarrow n_2 \) and \( \lambda_1 \rightarrow \lambda_2 \) in the equation for \( n_1 \), we have

\[ n_2 = 0, 1, 2, ... = \lambda_2 - \frac{1}{2} (|m| + 1) \]

Now, the sum of \( \lambda_1 \) and \( \lambda_2 \) must also be an integer, an integer that will turn out to be the principal quantum number. We therefore denote the sum \( \lambda_1 + \lambda_2 \) by \( n \).

\[ n = \lambda_1 + \lambda_2 = n_1 + n_2 + |m| + 1 \]

From this equation, together with \( \lambda_1 = \frac{1}{\alpha} (1 - \nu) \), \( \lambda_2 = \frac{\nu}{\alpha} \) and \( \alpha^2 = 2|E| \) we obtain the energy which, of course, must be the same as that obtained using
spherical coordinates. We have

\[ n = \lambda_1 + \lambda_2 = \frac{1}{\alpha} = \frac{1}{-2E} \]

where we have replaced \(|E|\) by \(-E\) since we seek the bound state energies. Finally, we have

\[ E = -\frac{1}{2n^2}; \quad \text{(au)} \]

Now, the degree of degeneracy cannot depend upon the coordinate system. To obtain the degree of degeneracy independent of the \(n^2\) determined in spherical coordinates we must count the parabolic eigenstates.

For \(\ell = 0\) there are \(n\) ways of choosing \(n_1\) and \(n_2\). For \(\ell > 0\) there are two ways of choosing \(m (=\pm \ell)\) and \(n - \ell\) ways of choosing \(n_1\) and \(n_2\). The degeneracy is therefore given by

\[
\text{degree of degeneracy} = n + 2 \sum_{\ell = 1}^{n-1} (n - \ell) = n + 2 \left[ n(n - 1) - \frac{n(n - 1)}{2} \right] = n^2
\]

where Gauss' trick is used to evaluate the sum of the first \(n\) integers.