d. Eigenfunctions of angular momentum operators

Consider now orbital angular momentum, so that \( j \to \ell \), a positive integer. We wish to now find the explicit functions \( |\ell \ell m\rangle \) in a specific coordinate system. We choose spherical coordinates \( r, \theta, \phi \) because we will be dealing with central potentials and the Schrödinger equation is separable in spherical coordinates for central potentials. The work that we have performed in obtaining the properties of the shift operators means that it is sufficient to find the eigenfunction \( |\ell \ell \rangle \), the eigenfunction with maximum \( z \)-component. We can generate all others by repeated application of \( J_\ell \).

We find \( |\ell \ell \rangle \) by solving the equation

\[
J_\ell |\ell \ell \rangle = 0
\]

We must at this point choose a representation in which to work. We can no longer use the abstract representation of operators. From the expressions for the Cartesian components of the vector operator for linear momentum

\[
p_x = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad \text{etc.}
\]

the definition of orbital angular momentum

\[
L = r \times p
\]

and the transformation equations from Cartesian to spherical coordinates we have

\[
L_x = -\frac{\hbar}{i} \left\{ \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right\}
\]

\[
L_y = -\frac{\hbar}{i} \left\{ \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right\}
\]

\[
L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}
\]

We may find the shift operators from their definitions, the above equations and the formula
\[ e^{\pm i\phi} = \cos \phi \pm i\sin \phi \]

We find that

\[ L_+ = \hbar e^{i\phi} \left( \frac{\partial}{\partial \theta} + \text{icot} \frac{\partial}{\partial \phi} \right) \]

\[ L_- = -\hbar e^{-i\phi} \left( \frac{\partial}{\partial \theta} - \text{icot} \frac{\partial}{\partial \phi} \right) \]

Since \( L_+ |\ell \rangle = 0 \) we have

\[ \hbar e^{i\phi} \left( \frac{\partial}{\partial \theta} + \text{icot} \frac{\partial}{\partial \phi} \right) \psi_{\ell}(\theta,\phi) = 0 \]

where

\[ \psi_{\ell}(\theta,\phi) = |\ell \rangle \]

Now we try separation of variables. Let

\[ \psi_{\ell}(\theta,\phi) = \Theta(\theta) \Phi(\phi) \]

We obtain

\[ \frac{\tan \theta}{\Theta} \frac{d\Theta}{d\theta} = -i \frac{1}{\Phi} \frac{d\Phi}{d\phi} \]

There are two important points about this equation to be noted.
1. The derivatives are total derivatives
2. The left side contains only \( \theta \)'s and the right side only \( \phi \)'s. Therefore each side must equal a constant, call it \( C \). Setting each side equal to \( C \) and integrating

\[ \Theta(\theta) \propto \sin^{C} \theta \quad \& \quad \Phi(\phi) \propto e^{iC\phi} \]

so that

\[ \psi_{\ell}(\theta,\phi) = N \sin^{C} \theta e^{iC\phi} \]
where $N$ is a normalization constant.

Now we must find $C$. We do this by requiring

$$L_{z} \psi_{\ell \ell}(\theta, \phi) = \ell \hbar \psi_{\ell \ell}(\theta, \phi)$$

and noting that another expression for $L_{z} \psi_{\ell \ell}(\theta, \phi)$ is

$$L_{z} \psi_{\ell \ell}(\theta, \phi) = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \psi_{\ell \ell}(\theta, \phi) = \frac{\hbar}{i} (iC) \psi_{\ell \ell}(\theta, \phi)$$

Comparison of the last two equations in boxes reveals that $C = \ell$.

**EXAMPLE**

Generate the eigenfunction $|\ell \ (\ell - 1)\rangle$.

Method:

$$L_{-} |\ell \ell \rangle = C^{-}_{\ell \ell} \hbar |\ell \ (\ell - 1)\rangle$$

$$= -\hbar e^{-i\phi} \left\{ \frac{\partial}{\partial \theta} - \frac{i}{\cot \theta} \frac{\partial}{\partial \phi} \right\} N \sin^{\ell} \theta e^{i \ell \phi}$$

$$= -N \hbar e^{-i\phi} \left\{ \ell \left[ \sin^{\ell-1} \theta \cos \theta \right] - i(i\ell) \cot \theta \sin^{\ell} \theta \right\} e^{i \ell \phi}$$

$$= -2N \hbar \ell \sin^{\ell-1} \theta \cos \theta e^{i(\ell-1)\phi}$$

Since

$$C^{-}_{\ell \ell} = \sqrt{\ell(\ell + 1) - \ell(\ell - 1)}$$

$$= \sqrt{2 \ell}$$

Therefore everything in the coefficient that is not $2^{1/2} \ell$ must belong to $\psi_{\ell, \ell-1}$,

$$\psi_{\ell, \ell-1}(\theta, \phi) = -\sqrt{2 \ell} N \left[ \sin^{\ell-1} \theta \right] \cos \theta e^{i(\ell-1)\phi}$$

Now, the expressions for $\psi_{\ell \ell}$ and $\psi_{\ell, \ell-1}$ are, of course, spherical harmonics which
are traditionally designated $Y_{\ell m}$. In fact, as comparison of the functions above with a table of spherical harmonics shows, we have actually generated $Y_{\ell \ell}$ and $Y_{\ell,\ell-1}$, although we have not considered normalization.