c. Raising and lowering operators

Manipulation of angular momentum operators is facilitated by introduction of the raising and lowering operators which are defined as

\[ J_+ = J_x + i J_y \quad \& \quad J_- = J_x - i J_y \]

Using these definitions and the commutation relations between the components of \( J \), we obtain the following relations.

\[
\begin{align*}
[J_+, J_z] &= [J_x, J_z] + i [J_y, J_z] = -i \hbar J_y - \hbar J_x = -\hbar (J_x + i J_y) \\
&= -\hbar J_+
\end{align*}
\]

and

\[
\begin{align*}
[J_, J_z] &= \hbar J_+ \\
&\quad \& \quad [J_+, J_-] = 2\hbar J_z
\end{align*}
\]

Further, since \( J^2 \) commutes with all components of \( J \), it also commutes with the raising and lowering operators \( J_+ \) and \( J_- \).

Now, we will use \( J_+ \) and \( J_- \) to find the eigenvalues and eigenfunctions of the angular momentum. Since \( J^2 \) and \( J_z \) commute, we will find simultaneous eigenfunctions of these two operators. Suppose that the eigenstates are distinguished by two quantum numbers. Call them \( n \) and \( m \) and denote the eigenfunctions by the ket \( |n m> \). If \( m \) is the quantum number associated with \( J_z \), we have

\[ J_z |n m> = m |n m> \]

(we insert the \( \hbar \) for future convenience because we know that angular momentum is quantized in units of \( \hbar \))

What do we know at this point about \( m \)? One thing that we know is that it must be real because \( J_z \) is a hermetian operator (it represents an observable). We do not, however, know that it is an integer – it might be a continuous function. Nor do we know the range of \( m \).

What do we know about \( J^2 \)? Operation on the eigenfunction \( |n m> \) must produce
\[ J^2|n \, m\rangle = \hbar^2 f(n,m)|n \, m\rangle \]

where \( f(n,m) \) is a dimensionless function. We allow for the possibility that it might depend on both quantum numbers, \( n \) and \( m \).

We first investigate restrictions on the relative magnitudes of \( f \) and the quantum numbers. We have

\[ <n \, m|J^2 - J_z^2|n \, m> = \left( f(n,m) - m^2 \right) \hbar^2 \]

and

\[ <n \, m|J^2 - J_z^2|n \, m> = <J_x^2> + <J_y^2> \geq 0 \]

Comparing the two expressions we see that

\[ f(n,m) \geq m^2 \]

Now apply the raising and lowering operators recalling that they commute with \( J^2 \) so that they cannot affect the magnitude of the angular momentum. We show this as follows.

Apply \( J_+ \) to \( J^2|n \, m\rangle = \hbar^2 f(n,m)|n \, m\rangle \) to obtain

\[ J_+\{J^2|n \, m\rangle\} = \hbar^2 f\{J_+|n \, m\rangle\} \]

which may be rewritten as

\[ J^2\{J_+|n \, m\rangle\} = \hbar^2 f\{J_+|n \, m\rangle\} \]

This shows that \( \{J_+|n \, m\rangle\} \) is an eigenstate of \( J^2 \) with the same eigenvalue as \( |n \, m\rangle \), that is, \( f(n,m) \). Therefore, \( J_+|n \, m\rangle \) has the same magnitude as \( |n \, m\rangle \). Furthermore, the same argument applies to \( J_- \).

How about \( J_z \)? The same proof does not work because \( J_+ \) and \( J_- \) do not commute.
with \( J_z \). Therefore, application of \( J_+ \) or \( J_- \) to \( |n m> \) must "shift" the value of \( m \), but leave the magnitude unchanged. But, to what eigenvalue of \( J_z \) does the eigenstate \( \{J_z|n m>\} \) correspond? We apply \( J_z \) to find out.

\[
J_z \{J_+|n m>\} = (J_z J_+ + J_+ J_z - J_+ J_z) |n m> 
\]

where the second and third terms were added and subtracted. We have

\[
J_z \{J_+|n m>\} = (J_z J_+ + J_+ J_z) |n m> 
\]

since \([J_z, J_+] = J_+\). Then

\[
J_z \{J_+|n m>\} = (J_+ + J_+ J_z) |n m> 
\]

so that \( \{J_+|n m>\} \) is an eigenstate of \( J_z \) with eigenvalue \((m + 1)\). \( J_+ \) raises the eigenvalue. We see then that the action of \( J_+ \) on \( |n m> \) is to produce an eigenstate of \( J^2 \) and \( J_z \) that is proportional to \( |n (m + 1)\rangle \). That is

\[
J_+ |n m> = \hbar C^+_{nm} |n (m + 1)\rangle 
\]

and

\[
J_- |n m> = \hbar C^-_{nm} |n (m - 1)\rangle 
\]

We can visualize the actions of \( J_+ \) and \( J_- \) graphically for the case in which \( J \) represents the angular momentum of a rotating body.
Now, $J_{+}$ and $J_{-}$ cannot change $|J|^2$. Also, $f(n,m) \geq m^2$. Therefore, we cannot apply $J_{+}$ and $J_{-}$ indefinitely. There must be maximum and minimum values that result from the shifting procedure. We must have

$$J_{+}|n_{\text{max}}> = 0$$

so that

$$J_{-}J_{+}|n_{\text{max}}> = 0$$

Now, expand $J_{-}J_{+}$.

$$J_{-}J_{+} = (J_{x} - iJ_{y})(J_{x} + iJ_{y}) = J_{x}^2 + J_{y}^2 + iJ_{x}J_{y} - iJ_{y}J_{x} = J^2 - J_{z}^2 + i[J_{x}, J_{y}] = J^2 - J_{z}^2 - \hbar J_{z}$$

so that

$$J_{-}J_{+}|n_{\text{max}}> = J^2 - J_{z}^2 - \hbar J_{z}$$

which leads to

$$J_{+}|n_{\text{max}}> = 0$$
\[ J^2 \ln |n \text{max}\rangle = (J_x^2 + \hbar J_z) \ln |n \text{max}\rangle = (m_{\text{max}}^2 \hbar^2 + \hbar^2 m_{\text{max}}) \ln |n \text{max}\rangle = m_{\text{max}} (m_{\text{max}} + 1) \hbar^2 \ln |n \text{max}\rangle \]

Therefore, we must have

\[ f(n, m_{\text{max}}) = m_{\text{max}} (m_{\text{max}} + 1) \]

Now, when \( J^- \) operates on \(|n \rangle \) it does not change the eigenvalue of \( J^2 \), but, it does lower \( m \) by 1.

\[ J^2 |m_{\text{max}} m\rangle = \hbar^2 m_{\text{max}} (m_{\text{max}} + 1) |m_{\text{max}} m\rangle \]

We see that the magnitude of the angular momentum is determined by the value of \( m_{\text{max}} \), a value which we do not as yet know. To investigate this we let \( m_{\text{max}} = j \) so that

\[ J^2 |j m\rangle = \hbar^2 j (j + 1) |j m\rangle \quad \text{where} \quad m = j, j - 1, ... \]

Now, what is the result of all of this manipulation? We have shown that \( j \) is the maximum value of \( m \), but not much else. What do we have left to show?

1. The nature of \( j \) and \( m \), e.g. integers? + or -?
2. Lower bound on \( m \). We know that it exists because \( m^2 \leq f \).

Of course we will find that \( m_{\text{min}} = -j \).

We begin by finding two expressions for the matrix elements of \( J^-J^+ \).

\[ J^-J^+ |j m\rangle = (J_x^2 - J_z^2 - \hbar J_z) |j m\rangle = \left( \hbar^2 j (j + 1) - m^2 \hbar^2 - m \hbar^2 \right) |j m\rangle = \hbar^2 \left( j (j + 1) - m (m + 1) \right) |j m\rangle \]

Now to find another expression for this matrix element we recall that
\[ J_+ |j m\> = \hbar C^+_{jm} |j (m + 1)\> \]
and
\[ J_- |j m\> = \hbar C^-_{jm} |j (m - 1)\> \]
so that
\[ <j m|J_+|j (m + 1)> = <j m|J_+^{\dagger}|j m>^* \]
Comparing the two expressions for the matrix element (boxes) we find that
\[ C^+_{jm} C^-_{j,m+1} = j(j + 1) - m(m + 1) \]
To complete the analysis we must find the connection between \( C^+_{j,m} \) and \( C^-_{j,m} \).

**EXERCISE**

Show that -
\[ <j m|J_-|j (m + 1)> = <j (m + 1)|J_+|j m>^* \]
i.e. prove that \( J_+ \) and \( J_- \), while not hermetian operators, are hermetian conjugates.

**SOLUTION TO THE EXERCISE**

\[ <j m|J_-|j (m + 1)> = <j m|J_x - iJ_y|j (m + 1)> \]
\[ = <j m|J_x|j (m + 1)> - i<j m|J_y|j (m + 1)> \]
\[ = <j (m + 1)|J_x|j m>* + i<j (m + 1)|J_y|j m>* \]
\[ = <j (m + 1)|J_x + iJ_y|j m>* \]

Now to find the connection between \( C^+_{j,m} \) and \( C^-_{j,m} \) we use the relation derived in the above exercise, i.e.
\[ <j m|J_-|j (m + 1)> = <j (m + 1)|J_+|j m>^* \]
We find that, since

\[ <j (m+1)|J \cdot l j m> = \hbar C_{jm}^+ <j (m+1)|j (m+1)> = \hbar C_{jm}^+ \]

and

\[ <j m)|J \cdot l j (m+1)> = \hbar C_{j (m+1)}^- <j mlj m> = \hbar C_{j (m+1)}^- \]

then

\[ C_{j, m+1}^- = C_{j, m}^+ * \]

But, we have already derived

\[ C_{jm}^+ C_{j, m+1}^- = j(j + 1) - m(m + 1) \]

so that, inserting the last equation in the box, we have

\[ C_{jm}^+ C_{j, m}^- * = j(j + 1) - m(m + 1) \]

Now we choose, for convenience, \( C_{jm}^+ \) to be real and positive. This leads to

\[ C_{jm}^+ = \sqrt{j(j + 1) - m(m + 1)} \]

and

\[ C_{jm}^- = C_{j, m-1}^+ * = \sqrt{j(j + 1) - m(m - 1)} \]

**EXAMPLE**
Evaluate the matrix element:
\[ <j (m+1)|J_x|j m> \]
SOLUTION

Express $J_x$ in terms of $J_+$ and $J_-$

$J_+ = J_x + jJ_y$

$J_- = J_x - jJ_y$

Adding and solving for $J_x$ we get

$J_x = \left(\frac{1}{2}\right)(J_+ + J_-)$

Then

$<j (m+1)|J_x|j m> = <j (m+1)|\left(\frac{1}{2}\right)(J_+ + J_-)|j m> = \left(\frac{1}{2}\right)\left(<j (m+1)|J_+|j m> + <j (m+1)|J_-|j m>\right) = \left(\frac{1}{2}\right)\left(C^+_{j m}h<j (m+1)|j (m+1)|j m> + C^-_{j m}h<j (m+1)|J_-|j (m-1)>\right) = \left(\frac{1}{2}\right)C^+_{j m}h = \left(\frac{1}{2}\right)\sqrt{j(j+1) - m(m+1)} h$

EXERCISE

Evaluate $<j (m+2)J_x^2|j m>$

Now that we have the coefficients, we may consider the effect of $J_-$ on the state with the lowest possible value of $m$, call it $m_{\text{min}}$. Clearly,

$J_-|j m_{\text{min}}> = 0$

and

$<j (m_{\text{min}} - 1)|J_-|j m_{\text{min}}> = 0$

Further, since

$|j m> = hC^-_{j m}|j (m - 1)>$

the above matrix element is also given by

$<j (m_{\text{min}} - 1)|J_-|j m_{\text{min}}> = hC^-_{j m}<j (m_{\text{min}} - 1)|j (m_{\text{min}} - 1)> = hC^-_{j m}$
Since

\[ C_{j}^{m} = \sqrt{j(j + 1) - m(m - 1)} \]

we have

\[ C_{j}^{m_{\text{min}}} = 0 = \sqrt{j(j + 1) - m_{\text{min}}(m_{\text{min}} - 1)} \]

so that \( m_{\text{min}} = -j \) and we have \(-j \leq m \leq +j\)

The symmetrical nature of this relationship has important consequences. To form such a ladder there are only two choices for \( j \):

1. \( j = \text{integer} \)
2. \( j = 1/2 - \text{integer} \)

Example: \( j = 2 \) \( \Rightarrow \) \( m = -2, -1, 0, +1, +2 \)

Example: \( j = 3/2 \) \( \Rightarrow \) \( m = -3/2, -1/2, +1/2, +3/2 \)

We will see that when boundary conditions are applied to spatial wave functions then \( j \) will have to take on integral values. Half-integral angular momenta do, however, exist. Electron spin is such an angular momentum, an "internal" or "intrinsic" angular momentum.

By convention, integral values of angular momentum are designated by \( \ell \) and \( m_{\ell} \), that is the quantum numbers \( j \rightarrow \ell \), and \( m \rightarrow m_{\ell} \). For spin angular momentum the convention is that \( j \rightarrow s \quad \& \quad m \rightarrow m_{s} \).
SUMMARY

\[ J^2 |j m_j> = \hbar^2 j(j+1) |j m_j> \]
\[ J_z |j m_j> = m_j \hbar |j m_j> \]
\[ <j (m_j + 1)|J_z|j m_j> = \hbar \sqrt{j(j+1) - m_j(m_j + 1)} \]
\[ <j (m_j - 1)|J_z|j m_j> = \hbar \sqrt{j(j+1) - m_j(m_j - 1)} \]

These results were derived using only hermiticity of the \( J_q \) and the commutation relations. What we do not have as yet are the eigenfunctions of the angular momentum operators.