b. Commutators

In most situations the angular momentum is a vector operator in three dimensions. Thus, a general angular momentum is given by

\[ J = J_x i + J_y j + J_z k \]

If the commutation relations between the components of \( J \) are given by

\[ [J_i, J_j] = i\hbar J_k \quad \text{et. cycle} \]

then \( J \) is defined as an angular momentum, irrespective of whether the classical counterpart of the operator represents angular motion. Nevertheless, we visualize such a motion and will refer to rotation when discussing angular momentum operators. Later on, we will encounter quantum mechanical operators for which the above commutation relations apply, but cannot be visualized classically. Of course, the most familiar of these is the "intrinsic" angular momentum – the spin.

The above commutator indicates that a rotating body can have only one of its components of angular momentum specified precisely. In keeping with accepted tradition, we choose \( J_z \) as this component. Then \( J_x \) and \( J_y \) cannot have sharply defined values. If we know \( J_z \) can we simultaneously know the magnitude of \( J \)? To answer this question we examine

\[ J^2 = J_x^2 + J_y^2 + J_z^2 \]

which is the square of the magnitude of the total angular momentum. This permits us to avoid the radials.

If \( J_z \) commutes with \( J^2 \) then these two operators can be specified simultaneously. This commutator is

\[ [J^2, J_z] = [J_x^2, J_z] + [J_y^2, J_z] + [J_z^2, J_z] \]

where the last term is obviously zero. The first term is

\[ [J_x^2, J_z] = J_x J_x J_z - J_z J_x J_x \]

Now add and subtract \( J_x J_z J_x \) to obtain
\[
\begin{align*}
[J_x^2, J_z] &= J_x J_x J_z + \langle J_x J_z J_x - J_z J_x J_z \rangle - J_z J_x J_x \\
&= J_x [J_x, J_z] + [J_x, J_z] J_x 
\end{align*}
\]

But
\[
[J_x, J_z] = -[J_z, J_x]
\]

which puts x, y & z in cyclic order. We then have
\[
\begin{align*}
[J_x^2, J_z] &= - \{ J_x i\hbar J_y + i\hbar J_y J_x \} \\
&= -i\hbar \{ J_x J_y + J_y J_x \}
\end{align*}
\]

The same operations on \([J_y^2, J_z]\) yields
\[
\begin{align*}
[J_y^2, J_z] &= + i\hbar \{ J_x J_y + J_y J_x \}
\end{align*}
\]

so that
\[
[J^2, J_z] = 0
\]

By symmetry we have
\[
[J^2, J_x] = 0 = [J^2, J_y]
\]

which shows that \(|J|\) can be specified together with any one of its components which we choose to be \(J_z\). Once, however, that we have chosen \(J_z\), \(J_x\) and \(J_y\) cannot be specified because
\[
[J_i, J_j] = i\hbar J_k \quad \text{et. cycle}
\]