1

Time-Dependent States in One Dimension

So far we have considered only eigenstates.

Quantum systems generally "live" in superpositions of eigenstates.

What is the relationship between these superpositions and a classical particle?

The de Broglie waves interfere constructively and destructively to form packet that represents a particle, a "wave packet".

1.1 Quantum Representation of Particles—Wave Packets

Because the TISE is a linear differential equation the wave function for any state may be written as a linear combination of eigenstates with the appropriate exponential time dependences.

Such an expansion is not limited to bound states.

In fact, particles need not be bound by *any* potential energy function.

We may use free particle eigenfunctions as a basis set to represent a particle, the k-states discussed above.

The framework for this superposition of k-states already exists.

It is the Fourier integrals which, including the time, are

$$\Psi\left(x,t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A\left(k,t\right)e^{ikx}dk$$

where $A\left(k\right)$ is the amplitude of the kth component of the wave packet. The factor $1/\sqrt{2\pi}$ is inserted for convenience.

Because we know the time dependence of the free particle eigenfunctions to be $e^{-iEt/\hbar}$ we write

$$\Psi\left(x,t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A\left(k\right)\left[e^{ikx}e^{-i\hbar k^{2}t/2m}\right]dk \qquad (1.1)$$

where we have replaced the free particle energy with

$$\begin{aligned}E &= \frac{p^2}{2m}\\ &= \frac{\hbar^2k^2}{2m}\end{aligned}$$

Return to the question of the actual speed of the particle that our packet is to represent. Write Equation 1.1 by substituting the $E = \hbar\omega$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega t)} dk$$

The wave frequency ω is, in general, a function of the wave number k. If it is assumed that ω is slowly varying and expand it in a Taylor series about some arbitrary value of $k = k_0$ for which $\omega = \omega_0$ we have

$$\omega(k) \approx \omega_0 + \left(\frac{d\omega}{dk}\right)_{k=k_0} (k-k_0) + ...$$

which, when inserted in Equation **??**, gives

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{ik_0(x-\omega_0 t/k_0)} \int_{-\infty}^{\infty} A(k) \exp\left\{ i \left[x - \left(\frac{d\omega}{dk}\right)_{k=k_0} t \right] (k-k_0) \right\} dk \tag{1.2}$$

From Equation 1.2 it can be seen that the wave function $\Psi(x,t)$ represents a plane wave propagating with velocity that is reshaped by the integral that multiplies it. The phase velocity is not the particle velocity.

The velocity of the centroid of the packet, called the group velocity, moves with constant velocity and, for a free particle of wave number k_0, is given by

$$\begin{aligned} v_g &= \left(\frac{d\omega}{dk}\right)_{k=k_0} \\ &= \left[\frac{d}{dk}\left(\frac{\hbar k^2}{2m}\right)\right]_{k=k_0} \\ &= \frac{\hbar k_0}{m} \\ &= \frac{p_0}{m} \end{aligned}$$

which is, indeed, the classical particle velocity.

Because the expansion of $\Psi(x,t)$ in Equation 1.1 is an expansion on free particle wave functions, the energy is simply the free particle energy, the kinetic energy of a particle of momentum $p = \hbar k$, that is, $p^2/2m$. We may therefore replace ω in Equation 1.1 using

$$\begin{aligned} E &= \hbar\omega \\ &= p^2/2m \\ &= \hbar^2 k^2/2m \end{aligned}$$

so that

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx} e^{-ik^2\hbar t/(2m)} dk \tag{1.3}$$

Relationship between the two functions, $\Psi(x, t=0)$ and $A(k)$.

The Fourier integral at $t = 0$ is [using the notation $\Psi(x, 0) = \psi(x)$]

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk \tag{1.4}$$

The function $\psi(x)$ is the Fourier transform of $A(k)$ while $A(k)$ is the Fourier transform of $\psi(x)$:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx \tag{1.5}$$

We may also write an expression for $A(k, t)$. It is

$$\begin{aligned} A(k,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,t) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} e^{-ik^2 \hbar t/(2m)} dx \end{aligned} \tag{1.6}$$

where $e^{-ikx} e^{-ik^2 \hbar t/(2m)}$ are the momentum space eigenfunctions.

Notice the symmetry between Equations 1.3 and 1.6.

$\psi(x)$ is the wave function in what referred to as coordinate space.

$A(k)$ is the wave function in k-space (momentum space)

Rather than writing the Fourier transform of $\psi(x)$ in terms of wave number k and $A(k)$ it is often useful to write it in terms of the momentum p and the momentum wave function $\phi(p)$ defined as

$$\begin{aligned} \phi(p) &\equiv \frac{1}{\sqrt{\hbar}} A(k) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx \end{aligned} \tag{1.7}$$

and

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-ip^2 t/(2m\hbar)} dx \tag{1.8}$$

$\phi(p)$ is also a wave function in momentum space. Inserting the $A(k) \equiv \sqrt{\hbar}\phi(p)$ into Equation 1.4 we obtain $\psi(x)$ in terms of $\phi(p)$:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{ipx/\hbar} dp \tag{1.9}$$

The symmetry between Equations 1.7 and 1.9 makes clear the reason for the factor of $1/\sqrt{\hbar}$ in the defining relation between $A(k)$ and $\phi(p)$. Rewrite Equation 1.3 in terms of the momentum we have

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{ipx/\hbar} e^{-ip^2 t/(2m\hbar)} dp \tag{1.10}$$

1.1.1 The Dirac δ-function

Replace $\phi(k)$ in Equation 1.4 with Equation 1.5 and interchange the order of integration:

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x')\,e^{-ikx'}dx'\right]e^{ikx}dk \\ &= \int_{-\infty}^{\infty}\psi(x')\,dx'\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik\left(x-x'\right)}dk\right] \end{aligned}$$

where x' is a dummy variable. The quantity in square brackets is defined to be the δ-function. It depends only on $(x-x')$ because the k integrates out.

By definition

$$\delta(x-x') \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik\left(x-x'\right)}dk \tag{1.11}$$

Now, the integral in Equation 1.11 is undefined as can be seen by converting the exponential to sines and cosines. What then is its meaning? In fact, the name δ-"function" is not proper. Mathematicians refer to such an entity as a distribution. No matter! We must use it in the context of its properties and its usage in quantum mechanics. In terms of the δ-function Equation **??** is

$$\psi(x) = \int_{-\infty}^{\infty}\delta(x-x')\,\psi(x')\,dx' \tag{1.12}$$

This equation illustrates one of the most important properties of the δ-function, the sifting property. According to this equation $\delta(x-x')$ sifts out the value of $\psi(x')$ at $x'=x$ and replaces the entire integral with $\psi(x'=x)$.

The properties of the δ-function are usually listed in terms of some arbitrary function $f(x)$ and a constant.

TABLE 1.1. Some properties of the Dirac delta-functions.

Mathematical operation	Name
$f(x_0) = \int_{-\infty}^{\infty}\delta(x-x_0)\,f(x)\,dx$	Sifting property
$\delta(-x) = \delta(x)$	Parity: even (if $x_0 = 0$)
$\int_{-\infty}^{\infty}\delta(x-x_0) = 1$	Normalization
$\delta(ax) = (1/\lvert a\rvert)\,\delta(x)$	None

1.2 Motion of a Wave Packet

To create the initial conditions, we imagine a particle that is initially subjected to a harmonic oscillator potential.

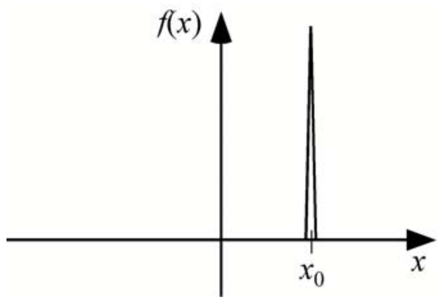

FIGURE 1.1. Schematic illustration of a the function $\delta(x - x_0)$. It is zero everywhere except at $x = x_0$ at which point it is infinitely high, but has zero width. Nonetheless, (believe it or not) the area under this spike is unity.

At $t = 0$ the motion is described by a momentum wave function $\Phi(p, 0)$ and a coordinate wave function $\Psi(x, 0)$ that are Fourier transforms of each other.

They are each Gaussians, but not the ground state of a harmonic oscillator.

Physically, we may imagine the particle is attached to a spring and oscillating, but not in any eigenstate of the harmonic oscillator Hamiltonian.

Investigate the fate of the packet under three different circumstances.

- Case I. The spring is cut and nothing is done thereafter (it is a free packet/particle).
- Case II. The spring is cut and a constant field is turned on at $t = 0$.
- Case III. Nothing is done. That is, the packet remains under the influence of the spring.

The wave functions are Fourier transforms of each other and are given by

$$\Psi(x, 0) = \frac{\sqrt{\beta}}{\pi^{1/4}} e^{-\beta^2 (x - x_0)^2 / 2} \cdot e^{i p_0 x / \hbar} \tag{1.13}$$

and

$$\Phi(p, 0) = \frac{1}{\pi^{1/4} \sqrt{\beta \hbar}} e^{-(p - p_0)^2 / 2 \beta^2 \hbar^2} \cdot e^{-i p x_0 / \hbar} \tag{1.14}$$

where we have used the constant β (rather than α) to emphasize that the system is *not* in an eigenstate. It is easily shown that for these wave packets $\Delta x_0 = 1/\left(\sqrt{2}\beta\right)$ and $\Delta p_0 = \beta\hbar/\sqrt{2}$. We may therefore write the wave functions at $t = 0$ in terms of the uncertainties Δx_0 and Δp_0:

$$\Psi(x,0) = \frac{1}{\pi^{1/4}}\left(\frac{1}{2^{1/4}\sqrt{\Delta x_0}}\right) e^{-(x-x_0)^2/4\Delta x_0^2} \cdot e^{ip_0 x/\hbar} \quad (1.15)$$

$$\Phi(p,0) = \frac{1}{\pi^{1/4}}\left(\frac{1}{2^{1/4}\sqrt{\Delta p_0}}\right) e^{-(p-p_0)^2/4\Delta p_0^2} \cdot e^{-ipx_0/\hbar} \quad (1.16)$$

Equations 1.15 and 1.16 illustrate an important property of Fourier transforms of Gaussian wave packets. Their uncertainties are equal in the sense that they occur in precisely the same form in each $\Psi(x,0)$ and $\Phi(p,0)$. An alternative way of saying this is that if $(x - x_0)$ and $(p - p_0)$ are measured in units of their respective uncertainties, then the functions have decreased by the same amount. For example, if $(x - x_0) = 2\Delta x$, then $\Psi(x,0)$ has decreased by one e-fold. In order for $\Phi(p,0)$ to decrease by one e-fold requires $(p - p_0) = 2\Delta p$.

It is actually more useful to have the absolute squares of $\Psi(x,0)$ and $\Phi(p,0)$ in terms of Δx_0 and Δp_0.

$$|\Psi(x,0)|^2 = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\Delta x_0}\right) e^{-(x-x_0)^2/2\Delta x_0^2} \quad (1.17)$$

$$|\Phi(p,0)|^2 = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\Delta p_0}\right) e^{-(p-p_0)^2/2\Delta p_0^2} \quad (1.18)$$

In what follows we wish to find the time dependence of the uncertainties. Inserting the time dependences into the last two equations we have

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\Delta x(t)}\right) e^{-(x-x_0)^2/2[\Delta x(t)]^2} \quad (1.19)$$

$$|\Phi(p,t)|^2 = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\Delta p(t)}\right) e^{-(p-p_0)^2/2[\Delta p(t)]^2} \quad (1.20)$$

1.2.1 Case I. The Free Packet/Particle

Cut the spring at a time such that $x_0 = 0$. The packet will then have nonzero average momentum p_0. The Gaussian packet in momentum space at $t = 0$ is therefore

$$\Phi(p,0) = \frac{1}{\pi^{1/4}\sqrt{\beta\hbar}} e^{-(p-p_0)^2/2\beta^2\hbar^2} \quad (1.21)$$

Let us first ask what we expect.

We expect the packet to propagate in the direction of p_0, $+x$ or $-x$.

We also expect the packet to change shape.

Being a free particle there can be no change in the momentum so that the initial spread in momentum Δp cannot change in time.

First we will find the wave function in coordinate space $\Psi(x,t)$. Inserting $\Phi(p,0) = \phi(p)$ in Equation 1.10 we have

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\pi^{1/4}\sqrt{\beta\hbar}} \int_{-\infty}^{\infty} e^{-(p-p_0)^2/\left(2\beta^2\hbar^2\right)} e^{ipx/\hbar} e^{-ip^2t/(2m\hbar)} dp \tag{1.22}$$

After some algebra we arrive at

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}} \frac{1}{\Delta x_0 \sqrt{1 + \frac{\hbar^2 t^2}{4m^2 \Delta x_0^4}}} \exp\left\{ - \left[\frac{(x - p_0 t/m)^2}{2\Delta x_0^2 \left(1 + \frac{\hbar^2 t^2}{4m^2 \Delta x_0^4}\right)} \right] \right\} \tag{1.23}$$

Because x and t occur in the combination $x - vt$, the probability packet travels with group velocity $v_g = p_0/m = \langle p \rangle / m$, the classical particle velocity.

Comparing Equation **??** with Equation 1.19, we see that the uncertainty as a function of time is given by

$$\Delta x(t) = \Delta x_0 \sqrt{1 + \left(\frac{\hbar t}{2\Delta x_0^2 m}\right)^2} \tag{1.24}$$

so that, in terms of $\Delta x(t)$, Equation 1.23 may be written more compactly as

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}} \frac{1}{\Delta x(t)} \exp\left\{ - \left[\frac{(x - p_0 t/m)^2}{2\Delta x(t)^2} \right] \right\} \tag{1.25}$$

It is seen that, in coordinate space, the packet spreads as it moves along.

On the other hand, this is a free particle so Δp must be independent of time.

Because the only time dependence in the momentum wave function is in the imaginary exponent, the time will not appear in the integrand of either $\langle p^2 \rangle$ or $\langle p \rangle$.

The time appears in Δx because x and x^2 must be changed to their momentum notation, derivatives, which operate on the time-dependent part of the imaginary exponent.

The uncertainty product $\Delta x \Delta p$, while initially its minimum value, grows with time.

The amplitude of the probability density decreases as indicated by the preexponential factor.

1.2.2 *Case II. The Packet/Particle Subjected to a Constant Field*

At $t = 0$ the Gaussian packet is subjected to a constant force φ.

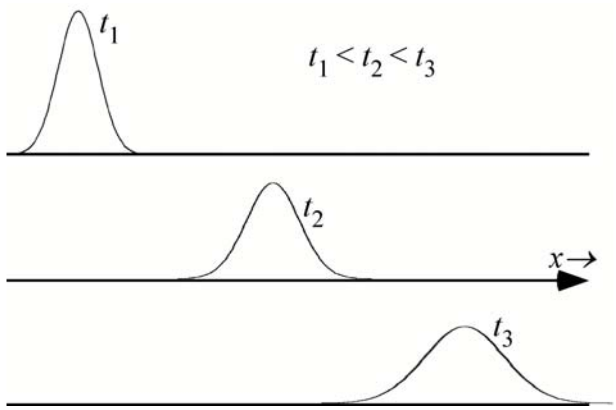

FIGURE 1.2. A free Gaussian wave packet shown at three different times. Note that the width of the packet increases in time, but the area under the curve remains constant.

How could such a situation arise? If the particle of mass m carries an electrical charge and if it is in a region of constant electric field, then the force is the product of the charge and the electric field.

It would also occur if a particle oscillating on a hanging spring were suddenly set free by cutting the spring. After cutting the spring the particle is subjected to the constant gravitational force.

Without specifying the origin of the force we may write the potential as

$$U(x) = -\varphi x; \quad -\infty < x < \infty \tag{1.26}$$

To simplify the mathematics we take the Gaussian packet to be one for which the average momentum and average displacement are zero. In momentum space the initial packet is described by

$$\Phi(p,0) = \frac{1}{\pi^{1/4}\sqrt{\beta\hbar}} e^{-p^2/2\beta^2\hbar^2} \tag{1.27}$$

The TDSE with the potential energy of Equation 1.26 can be solved exactly in coordinate space (see Section **??**), but for the present purpose it is convenient to write the TDSE in momentum space. Using Equation **??** to replace $x \rightarrow (i\hbar)\,\partial/\partial p$ in the TDSE with a linear potential, we have

$$\frac{p^2}{2m}\Phi(p,t) - i\hbar\varphi\frac{\partial\Phi(p,t)}{\partial p} = i\hbar\frac{\partial\Phi(p,t)}{\partial t} \tag{1.28}$$

This partial differential equation may be solved by making the substitution

$$\Phi(p,t) = \Theta(p')\,f(p) \quad \text{where} \quad p' = p - \varphi t \tag{1.29}$$

which leads to a differential equation for the function $f(p)$

$$\varphi \frac{df(p)}{dp} = \frac{p^2}{2m(i\hbar)} f(p) \tag{1.30}$$

the solution to which is

$$f(p) = \exp\left(-\frac{ip^3}{6m\hbar\varphi}\right) \tag{1.31}$$

so that

$$\Phi(p,t) = \Theta(p - \varphi t) \exp\left(-\frac{ip^3}{6m\hbar\varphi}\right) \tag{1.32}$$

where $\Theta(p - \varphi t)$ is *any* function of $(p - \varphi t)$ (see Problem **??**). Initial conditions fix $\Theta(p - \varphi t)$.

To determine the $\Theta(p - \varphi t)$ that corresponds to the wave packet in Equation 1.27 we set $t = 0$ in Equation 1.32 and equate the result to the wave function representing the initial Gaussian wave packet, Equation 1.27. This permits determination of $\Theta(p)$ which can immediately be converted to $\Theta(p - \varphi t)$ because this function can contain p and t in only the combination $(p - \varphi t)$ (see Problem **??**). We obtain

$$\Theta(p - \varphi t) = \left(\frac{1}{\pi^{1/4}\sqrt{\beta\hbar}}\right) \exp\left(-\frac{(p - \varphi t)^2}{2\beta^2\hbar^2} + \frac{i(p - \varphi t)^3}{6m\hbar\varphi}\right) \tag{1.33}$$

Substituting Equation 1.33 into Equation 1.32 we obtain the time-dependent wave function in momentum space for a Gaussian wave packet:

$$\Phi(p,t) = \left(\frac{1}{\pi^{1/4}\sqrt{\beta\hbar}}\right) \exp\left(-\frac{(p - \varphi t)^2}{2\beta^2\hbar^2}\right) \exp\left[i\left(\frac{(p - \varphi t)^3 - p^3}{6m\hbar\varphi}\right)\right] \tag{1.34}$$

and the probability density in momentum space is

$$|\Phi(p,t)|^2 = \left(\frac{1}{\sqrt{\pi}\beta\hbar}\right) \exp\left[-\frac{(p - \varphi t)^2}{\beta^2\hbar^2}\right] \tag{1.35}$$

or in terms of $\Delta p_0 = \beta\hbar/\sqrt{2}$

$$|\Phi(p,t)|^2 = \left(\frac{1}{\sqrt{2\pi}\Delta p_0}\right) \exp\left[-\frac{(p - \varphi t)^2}{2\Delta p_0^2}\right] \tag{1.36}$$

Comparing Equation 1.35 with Equation 1.20 reveals that

$$\Delta p(t) = \Delta p_0 \tag{1.37}$$

which contains no time dependence. Thus, as for the free particle Gaussian wave packet, this packet does not spread in momentum. Why is this? After all, there is a force applied. The force is, however, constant so all momentum components are affected equally. The packet moves as a unit in momentum space, but it does not spread.

It is straightforward to extract the time-dependent expectation values $\langle x(t)\rangle$ and $\langle p(t)\rangle$ (see Problem **??**). We obtain

$$\langle x(t)\rangle = \frac{\varphi t^2}{2m} \quad \text{and} \quad \langle p(t)\rangle = \varphi t \tag{1.38}$$

both of which are consistent with the Ehrenfest equations. Note that $\langle x(t)\rangle$ has the familiar t^2 dependence of any particle under the influence of a constant force because, by Newton's second law, the acceleration is φ/m. The expectation value of the momentum is indeed Newton's second law because the force is the time rate of change of the (average) momentum.

Consider now the uncertainty in position $\Delta x(t)$. We already know $\langle x(t)\rangle$ so one method of obtaining $\Delta x(t)$ is to compute $\left\langle x(t)^2\right\rangle$ using the momentum space wave function, Equation 1.34, and replacing x^2 in the integral with $\hbar^2 d^2/dp^2$. Alternatively, we could obtain $\Psi(x,t)$ by performing a Fourier transform on the momentum wave function, squaring, and identifying $\Delta x(t)$ by comparing with Equation 1.19. The Fourier transform yields

$$\Psi(x,t) = \frac{1}{\pi^{1/4}}\sqrt{\frac{\beta}{\gamma}}\exp\left[\frac{i\varphi t}{\hbar}\left(x-\frac{\varphi t^2}{6m}\right)\right]\cdot\exp\left\{-\frac{\left[x-\varphi t^2/(2m)\right]^2}{\left(2\gamma/\beta^2\right)}\right\} \tag{1.39}$$

where, defining $t_0 = m/\left(\hbar\beta^2\right)$ as in Equation **??**,

$$\gamma = 1+\frac{it}{t_0} \quad \text{and} \quad t_0 = \frac{m}{\hbar\beta^2} = \frac{2m}{\hbar}\Delta x_0^2 \tag{1.40}$$

The probability density in coordinate space is then

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{|\gamma|^2/\beta^2}}\right)\exp\left\{-\frac{\left[x-\varphi t^2/(2m)\right]^2}{\left(|\gamma|^2/\beta^2\right)}\right\} \tag{1.41}$$

Comparing Equation 1.41 with Equation 1.19 we see that

$$2\Delta x(t)^2 = \frac{|\gamma|^2}{\beta^2} \tag{1.42}$$

so that in terms of $\Delta x(t)$ we have

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\Delta x(t)}\right)\exp\left\{-\frac{\left[x-\varphi t^2/(2m)\right]^2}{2\Delta x(t)^2}\right\} \tag{1.43}$$

where, recalling that $\Delta x_0 = 1/\left(\sqrt{2}\beta\right)$

$$\Delta x(t) = \Delta x_0 \left(1 + \frac{t^2}{t_0^2}\right)^{1/2} \tag{1.44}$$

which is identical to Equation 1.24, again a consequence of the constant force being applied.

1.2.3 Case III. The Packet/Particle Subjected to a Harmonic Oscillator Potential

Assume that we have a Gaussian wave packet that is a linear superposition of harmonic oscillator eigenstates and that $\Psi(x, 0)$ is known.

Choose an initial wave function of the form

$$\Psi(x, 0) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\alpha^2 (x - x_0)^2 / 2} \tag{1.45}$$

where here $\alpha = \sqrt{m\omega/\hbar}$, the same constant that appears in the eigenfunctions of the harmonic oscillator.

The nonzero average displacement assures us that Equation 1.45 is not an eigenfunction of the harmonic oscillator Hamiltonian. Of course, it may be expanded upon the complete set of harmonic oscillator eigenfunctions.

This is a Gaussian distribution with average displacement x_0 and zero initial momentum.

Classically this is equivalent to pulling the particle to $x = x_0$ and releasing it with no initial momentum. Such a state is sometimes referred to as a displaced ground state.

Remember, the particle remains under the influence of the potential energy $U(x) = \frac{1}{2}kx^2$.

Our goal is to find the function $\Psi(x, t)$ so that we may determine the time dependence of the probability distribution $|\Psi(x, t)|^2$.

No need to determine the momentum space wave function so we do not require any Fourier transforms.

Expanding $\Psi(x, t)$ on the complete set of harmonic oscillator eigenfunctions $\psi_n(x) e^{-i(E_n/\hbar)t}$ we have

$$\Psi(x, t) = \sum_{n=1}^{\infty} a_n \psi_n(x) e^{-i(E_n/\hbar)t} \tag{1.46}$$

To find the a_n we have to multiply both sides by $\Psi(x, 0)$, Equation 1.45.

In this particular case, however, there is an easier way using the generating function for the Hermite polynomials.

For the Hermite polynomials the generating function is

$$e^{2\mu\xi - \mu^2} = \sum_{n=0}^{\infty} \frac{H_n(\xi)\,\mu^n}{n!} \tag{1.47}$$

Using the scaled distance $\xi = \alpha x$ the initial packet is

$$\Psi(\xi, 0) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-(\xi-\xi_0)^2/2}$$

Letting $\xi_0 = 2\mu$ so that

$$\begin{aligned}\Psi(\xi, 0) &= \frac{\sqrt{\alpha}}{\pi^{1/4}} \exp\left[-\frac{\xi^2}{2} + 2\mu\xi - 2\mu^2\right] \\ &= \frac{\sqrt{\alpha}}{\pi^{1/4}} \exp\left[-\frac{\xi^2}{2} - \mu^2 + 2\mu\xi - \mu^2\right] \\ &= \frac{\sqrt{\alpha}}{\pi^{1/4}} \exp\left[-\left(\frac{\xi^2}{2} + \mu^2\right)\right] \cdot \exp\left(2\mu\xi - \mu^2\right) \qquad (1.48)\end{aligned}$$

The last term is just the generating function of the Hermite polynomials! We may therefore replace it using Equation 1.47:

$$\begin{aligned}\Psi(\xi, 0) &= \frac{\sqrt{\alpha}}{\pi^{1/4}} \exp\left[-\left(\frac{\xi^2}{2} + \mu^2\right)\right] \sum_{n=0}^{\infty} \frac{H_n(\xi)\,\mu^n}{n!} \\ &= \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\mu^2} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \left\{e^{-\xi^2/2} H_n(\xi)\right\} \qquad (1.49)\end{aligned}$$

The terms in the brackets in Equation 1.49 are the harmonic oscillator eigenfunctions.

Comparing Equation 1.49 with Equation 1.46 at $t = 0$ we see that we have "accidentally" calculated the expansion coefficients, the a_n.

To include the time in the wave function multiply each harmonic oscillator eigenfunction in the summation by the appropriate exponential.

$$\begin{aligned}\Psi(\xi, t) &= \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\mu^2} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \left\{e^{-\xi^2/2} H_n(\xi)\right\} \exp\left[-i\left(n + \frac{1}{2}\right)\omega t\right] \\ &= \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\mu^2} e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \left\{e^{-\xi^2/2} H_n(\xi)\right\} e^{-in\omega t}\end{aligned}$$

Removing $e^{-\xi^2/2}$ from the summation and regrouping the terms we have

$$\Psi(\xi, t) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\mu^2} e^{-i\omega t/2} e^{-\xi^2/2} \sum_{n=0}^{\infty} \left[\frac{\left(\mu e^{-i\omega t}\right)^n}{n!} H_n(\xi)\right] \qquad (1.50)$$

The summation is the generating function for the Hermite polynomials with $\mu \to \mu e^{-i\omega t}$ as is easily seen from Equation 1.47. That is,

$$\sum_{n=0}^{\infty} \frac{H_n(\xi)\left(\mu e^{-i\omega t}\right)^n}{n!} = \exp\left[2\xi\mu e^{-i\omega t} - \left(\mu e^{-i\omega t}\right)^2\right]$$

After substituting $\mu = \xi_0/2$, Equation 1.50 becomes

$$\Psi(\xi, t) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-i\omega t/2} \exp\left[-\left(\frac{\xi^2}{2} + \frac{\xi_0^2}{4}\right)\right] \cdot \exp\left[\xi_0 \xi e^{-i\omega t} - \frac{\xi_0^2}{4} e^{-2i\omega t}\right]$$

In terms of sines and cosines, we have

$$\Psi(\xi, t) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-i\omega t/2} \exp\left[-\frac{1}{2}\left(\xi^2 + \frac{\xi_0^2}{2}(1 + \cos 2\omega t) - 2\xi_0 \xi \cos \omega t\right)\right]$$
$$\times \exp\left[\frac{i}{2}\left(\frac{\xi_0^2}{2} \sin 2\omega t - 2\xi_0 \xi \sin \omega t\right)\right] \tag{1.51}$$

and the time-dependent probability density is

$$|\Psi(\xi, t)|^2 = \frac{\alpha}{\sqrt{\pi}} \exp\left\{-\left[\xi^2 + \frac{\xi_0^2}{2}(1 + \cos 2\omega t) - 2\xi_0 \xi \cos \omega t\right]\right\}$$
$$= \frac{\alpha}{\sqrt{\pi}} \exp\left[-(\xi - \xi_0 \cos \omega t)^2\right] \tag{1.52}$$

In terms of the coordinate x,

$$|\Psi(x, t)|^2 = \frac{\alpha}{\sqrt{\pi}} \exp\left[-\alpha^2 (x - x_0 \cos \omega t)^2\right] \tag{1.53}$$

Equation 1.53 shows that the wave packet oscillates about $x = 0$ so the expectation value of position as a function of time is

$$\langle x(t) \rangle = x_0 \cos \omega t \tag{1.54}$$

Comparison with Equation 1.19 shows that the uncertainty in position is

$$\Delta x(t) = \frac{1}{\sqrt{2}\alpha} = \Delta x_0 \tag{1.55}$$

which is time-independent. The packet oscillates without any change in shape!

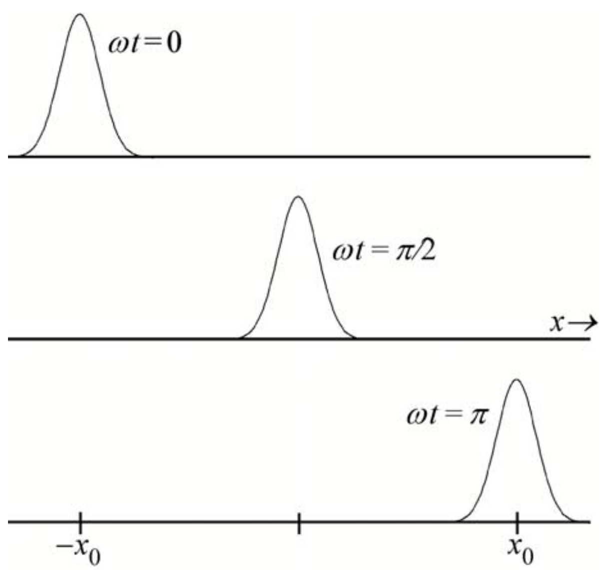

FIGURE 1.3. A Gaussian wave packet under the influence of a harmonic oscillator potential shown at three different times. Note that the shape of the packet does not change.