

Classical view of the Stark effect in hydrogen atoms

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The unique properties of the hydrogen atom, especially those that result from the dynamic symmetry of the $1/r$ potential, are presented from both classical and quantal points of view. Using these properties as a starting point, the response of a classical hydrogen atom, an electron executing a Keplerian ellipse about a proton, to a weak electric field is described. Because the field is weak, the elliptical orbit may be treated as a dynamical element itself, variations in its shape and orientation depicting the classical Stark effect. The behavior of this orbit is then correlated with the quantum mechanical description of the first-order Stark effect in hydrogen, a subject that is included in most introductory courses in quantum mechanics.

I. INTRODUCTION

Although the effect of an external electric field on an excited hydrogen atom is a subject that is included in most introductory quantum mechanics texts, it is usually done so in order to illustrate degenerate perturbation theory. The Stark effect has thus been relegated to a minor role, especially in comparison to the Zeeman effect, in undergraduate and graduate courses on quantum mechanics. This is unfortunate because in depth study of this subject can lead the student to a deeper understanding of the hydrogen atom as well as a greater appreciation for the special symmetry of the $1/r$ potential.

Because the form of the potential for the hydrogen atom and for planetary systems is identical, it is instructive to describe the Stark effect in hydrogen classically. Comparison of the classical and quantum descriptions can then provide insight to the properties of hydrogen and of classical Keplerian orbits. Moreover, the correspondence between classical and quantum quantities and concepts can also be elucidated by such a comparison.

In order to make these comparisons it is sufficient to formulate the classical Stark effect in hydrogen using fairly elementary methods. We assume that the Stark field is weak and ignore radiation by an accelerated charge, Amperian magnetic fields, and relativistic effects. It is therefore assumed that, in the absence of the Stark field, the motion of the electron is governed solely by an attractive $1/r$ potential. We are then able to use the well-known parameters of Keplerian motion,¹ together with vector analysis and the relationships given by Hamiltonian's equations, to derive equations that permit the classical motion to be visualized. Because the Stark field is weak, the classical Stark effect may be viewed as the response of the electronic orbit to application of the field.²

In the course of the derivations we often substitute quantum mechanical parameters into the classical equations to facilitate the comparison. For example, the angular momentum is replaced by the quantity $l\hbar$, where l is the angular momentum quantum number. To simplify the formulation, however, we use atomic units in which the electronic charge e , the electronic mass m_e , and \hbar are all set to unity. Table I is a listing of some pertinent quantities in this system of units including parameters for Keplerian orbits in terms of hydrogenic quantities.

II. THE HYDROGEN ATOM WITH NO EXTERNAL FIELDS

It is well known that the accidental degeneracy of the hydrogen atom that manifests itself in a proper quantum mechanical treatment is due to the dynamical symmetry³ of the $1/r$ potential. This symmetry is also responsible for the fact that the Schrödinger and Hamilton–Jacobi equations for the hydrogen atom may be separated in either spherical coordinates, as can be done for *any* central potential, and parabolic coordinates.⁴ Separation of the Schrödinger equation in spherical coordinates is accomplished using the mutually commuting operators H the Hamiltonian (total energy), L^2 the square of the angular momentum, and L_z the z component of L . These operators commute and, classically, they represent conserved quantities.⁵ Classically of course both L_x and L_y are also conserved, but the uncertainty principle precludes simultaneous determination of all three components of L in a quantum mechanical treatment. Only one component can be specific and it is traditional to choose L_z as the operator that com-

Table I. Hydrogenic quantities in atomic units and parameters for Keplerian orbits in terms of these hydrogenic quantities. In atomic units the mass of the electron, the electronic charge, and \hbar are all set to unity. n is the principal quantum number, l is the angular momentum quantum number, and m the magnetic quantum number ($-l < m < l$), but, for the classical quantities they are continuously variable.

Quantity	Value
Energy	$E = -1/2n^2$
Semimajor axis	$a = n^2$
Seminor axis	$b = nl$
Orbital period	$\tau = 2\pi a^{3/2}$ $= 2\pi n^3$
Orbital frequency	$\omega_n = 1/n^3$
Orbital eccentricity	$\epsilon = \sqrt{1 - l^2/n^2}$ $= \mathbf{A} $
Pericenter (from focus)	$r_{\min} = n^2(1 - \epsilon)$ $= l^2/(1 + \epsilon)$
Apoecenter (from focus)	$r_{\max} = n^2(1 + \epsilon)$ $= l^2/(1 - \epsilon)$

mates with \mathbf{H} and \mathbf{L}^2 . For the Coulomb and gravitational potentials there is however an additional conserved quantity,⁶ the Lenz vector⁷ \mathbf{A} .

The general definition, irrespective of the nature of the potential function, is given, in atomic units, by⁶

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \hat{\mathbf{r}}, \quad (1)$$

where \mathbf{p} is the linear momentum, \mathbf{L} the angular momentum, and $\hat{\mathbf{r}}$ the unit vector in the direction of the displacement vector \mathbf{r} . We note that this definition is not universal,⁶ but it is always given as some linear combinations of $\mathbf{p} \times \mathbf{L}$ and $\hat{\mathbf{r}}$. Since the quantity $\mathbf{A} \cdot \mathbf{L} \equiv 0$, the Lenz vector is always perpendicular to the angular momentum. This is true whether or not either quantity is a constant of the motion.

Although the definition of \mathbf{A} is not particularly revealing, it is easily shown that, classically, \mathbf{A} is a constant of the motion for the Coulomb and gravitational potentials, i.e., $d\mathbf{A}/dt = 0$. In addition, because we are working in atomic units, the magnitude of \mathbf{A} is equal to the eccentricity of the Keplerian ellipse. Further, \mathbf{A} as defined above, points in the direction of the pericenter of the elliptical orbit of bound Keplerian systems. Therefore, while the central nature of the potential requires constancy of \mathbf{L} , thus specifying the plane to which the motion is confined, the constancy of \mathbf{A} specifies the orientation of the closed elliptical path of the particle, electron, or planet.

If the potential is only slightly different from $1/r$, then $d\mathbf{A}/dt \neq 0$ and the ellipse precesses about the force center leading to an open trajectory. This phenomenon is of course the cause of the famous advance of the perihelion of the orbit of the planet Mercury, a problem that occupies a distinguished position in the history of physics. If the potential is significantly different from $1/r$, then, in general, the trajectory for a bound state is still open, but bears little resemblance to an ellipse or even a precessing ellipse. These open trajectories are such that the particle covers all points in the plane lying between the maximum and minimum values of the polar coordinate r . Figure 1 shows typical motion for a pure $1/r$ potential and motion for a potential that is slightly non-Coulombic. We have chosen to add a term proportional to $1/r^4$ as the non-Coulombic portion of

the potential for this illustration. Since it is central in nature, angular momentum is conserved and the motion is confined to a plane.

The total energy of a particle in Keplerian orbit depends only on the value of the semimajor axis and not on the magnitude of the angular momentum. The value of the semiminor axis, and thus the eccentricity of the elliptical orbit, does however depend on the angular momentum. The (classical) energy degeneracy, which is a consequence of the special symmetry of the $1/r$ potential, is, as might be suspected, related^{3,5} to the accidental degeneracy that appears in the quantum mechanical treatment of the hydrogen atom.

These degeneracies, classical and quantum mechanical, come about because the dynamic symmetry of the $1/r$ potential imposes an additional constant of the motion \mathbf{A} , which leads to a fourth mutually commuting quantum mechanical operator. Only three of these operators are required to solve the Schrödinger equation. If, as is commonly done in introductory texts, \mathbf{H} , \mathbf{L}^2 , and \mathbf{L}_z are chosen, then the Schrödinger equation can be separated in spherical coordinates as it can for any central potential. If, however \mathbf{L}^2 is replaced by a component of \mathbf{A} , usually taken to be \mathbf{A}_z , then the separation of variables can be accomplished in parabolic coordinates.^{3,4}

The eigenfunctions obtained from separation of the Schrödinger equation in spherical coordinates are characterized by the quantum numbers n , l , and m . These eigenfunctions, frequently referred to as angular momentum eigenfunctions, correspond to stationary states in which the energy, magnitude of the angular momentum and z -component of the angular momentum have definite values. The energy is given by the famous Rydberg formula

$$E = -1/2n^2. \quad (2)$$

Note that, as in the case of the classical Kepler problem, the energy is independent of the angular momentum.

Separation of the Schrödinger equation in parabolic coordinates employs \mathbf{H} , \mathbf{L}_z , and \mathbf{A}_z as the mutually commuting operators. This leads to a different set of eigenfunctions, characterized by a different set of quantum numbers, n_1 , n_2 , and m , which are related to the principal quantum number by

$$n = n_1 + n_2 + |m| + 1. \quad (3)$$

The quantum number m is the same as that in the angular momentum eigenfunction because \mathbf{L}_z is one of the commuting operators. The energy of these stationary states must of course be the same as that of the angular momentum eigenstates and is given by Eq. (2). The parameter associated with the third of the commuting operators, the z component of the Lenz vector, is given by

$$\mathbf{A}_z = (n_1 - n_2)/n. \quad (4)$$

Because parabolic coordinates are well suited to treatment of the Stark effect in hydrogen the quantity $(n_1 - n_2)$ is sometimes referred to as the electric quantum number k . Examination of Eq. (3) shows that, for fixed $|m|$, k changes in increments of 2.

III. THE HYDROGEN ATOM IN AN EXTERNAL ELECTRIC FIELD

For a hydrogen atom in a constant electric field \mathbf{F} the potential energy is given by

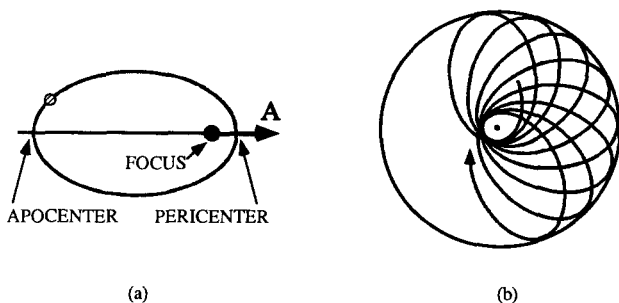


Fig. 1. Classical trajectories of particles under the influence of two different potentials. (a) An attractive $1/r$ potential, where r is the distance from the force center, that leads to a closed Keplerian ellipse with the force center at one focus of the ellipse. The Lenz vector \mathbf{A} , as defined in the text, is constant and points in the direction of the pericenter as shown. (b) The sum of an attractive $1/r$ potential and an attractive $1/r^4$ potential. The $1/r^4$ term causes the Keplerian ellipse produced by the $1/r$ potential to precess about the force center; the strength of the $1/r^4$ term has been adjusted to illustrate this precession. Because, in this case, the non-Coulombic term depends only on r , angular momentum is conserved and the motion is confined to a plane.

$$V(r, \theta) = -1/r + Fr \cos \theta \quad (5)$$

in spherical coordinates where F is taken to define the z direction. In parabolic coordinates for which

$$\xi = r + z, \quad \eta = r - z, \quad \phi = \tan^{-1}(y/x)$$

and

$$r^2 = x^2 + y^2 + z^2$$

this potential energy is

$$V(\xi, \eta, \phi) = -2/(\xi + \eta) + 1/2[F(\xi - \eta)], \quad (6)$$

The reason that parabolic coordinates are well suited for treatment of the Stark effect is that, even with the noncentral electric field term in the Hamiltonian, the Schrödinger equation can be separated. This cannot be accomplished in spherical coordinates. Upon solving in parabolic coordinates the energy levels are given, to first order, by⁴

$$E = -1/(2n^2) + (3/2)nkF. \quad (7)$$

The first term is the hydrogenic energy in field-free space, while the second represents the first-order (in F) effect of the electric field. The fact that there is a nonzero first-order term shows that excited states of hydrogen, in general, possess permanent electric dipole moments. This property is often ignored in introductory quantum mechanics courses because it does not manifest itself when treatment of the hydrogen atom is restricted to spherical coordinates. The parabolic eigenfunctions, however, lead to a physical picture of the permanent electric dipole moment of excited states of hydrogen.⁸ While the electron probability density, $\psi^*\psi$, is symmetric about the origin for angular momentum stationary states, in general $\psi^*\psi$ is skewed with respect to the origin for parabolic stationary states. Note that, as a result of the accidental degeneracy, a parabolic eigenstate may be represented as a linear combination of angular momentum eigenstates of the same energy; the reverse is of course also true.

Since the energy of an electric dipole \mathbf{d} in a field \mathbf{F} is given by $-\mathbf{d} \cdot \mathbf{F}$, it is clear that the coefficient of F , $(3/2)nk$, in Eq. (7) is the component of the dipole moment in the direction of the field. Further, because adjacent Stark states linked to a given field-free state differ in energy by $(3/2)nF$, the orientation of the dipole moment is quantized in the same way as is the angular momentum.

The general asymmetry of the electron density for parabolic stationary states makes it clear that these states do indeed have (time-averaged) permanent electric dipole moments. This corresponds classically^{9,10} to the Keplerian orbit shown in Fig. 1(a). As for *any* central potential the speed of the electron is governed by Kepler's second law, the law of equal areas. Thus near apocenter the electron is moving more slowly than near pericenter so that the *classical* electron density is skewed. The overall asymmetry results because the Keplerian ellipse is fixed in space, a consequence of the constancy of \mathbf{A} . There are, however, some excited states of hydrogen that do not have permanent electric dipole moments. Their classical analogs are circular Keplerian orbits. Quantum mechanically this corresponds to angular momentum eigenstates for which the azimuthal quantum number $l = n - 1$ or parabolic eigenstates for which the electric quantum number $k = 0$, i.e., $n_1 = n_2$.

It is in fact possible to calculate the average value of the dipole moment of an electron in Keplerian orbit about a proton. We begin with the equation of motion¹ in polar coordinates r and ϕ for a particle of energy E and reduced mass μ subjected to a central potential $V(r)$:

$$\phi(r) = \int \frac{(l/r^2) dr}{\{2\mu[E - V(r) - l^2/2\mu r^2]\}^{1/2}}$$

If $V(r) = -1/r$ and $\mu = 1$, the Coulomb potential and electronic reduced mass in atomic units, integration gives the orbit of the particle, a conic section with one focus at the origin, the equation of which is¹

$$l^2/r = 1 + \epsilon \cos \phi, \quad (8)$$

where ϵ , the eccentricity, is

$$\epsilon = \sqrt{1 - l^2/n^2} \quad (9)$$

and $\phi = 0$ is measured from the pericenter. Note that our interest is in $E < 0$, in which case the particle is bound and the orbit is a Keplerian ellipse. The quantities n and l in Eq. (9) refer to the classical energy, $E = -1/2n^2$, and the classical angular momentum, respectively; the maximum value of l is n .

By symmetry, the dipole moment points toward the pericenter, in the direction of \mathbf{A} . Thus

$$\langle \mathbf{d} \rangle = \langle r \cos \phi \rangle (\mathbf{A}/|\mathbf{A}|) \quad (10)$$

and the magnitude of the dipole moment is given by

$$|\langle d \rangle| = \frac{1}{\tau} \int_0^\tau r \cos \phi dt = \frac{1}{\tau} \int_0^\tau r \cos \phi \left(\frac{1}{\dot{\phi}}\right) d\phi, \quad (11)$$

where $\dot{\phi} = (d\phi/dt)$. Note that $\langle \mathbf{d} \rangle = -\langle \mathbf{r} \rangle$ in atomic units. Using Eq. (11) and conservation of angular momentum, $r^2 \dot{\phi} = l$, we obtain

$$|\langle d \rangle| = \frac{l^5}{2\pi n^3} \int_{-\pi}^\pi \frac{\cos \phi d\phi}{(1 + \epsilon \cos \phi)^3}, \quad (12)$$

which may be integrated by noting that the integrand can be written as

$$\left(-\frac{1}{2}\right) \frac{d}{d\epsilon} (1 + \epsilon \cos \phi)^{-2}.$$

Upon interchanging the order of the operations and integrating we obtain

$$|\langle \mathbf{d} \rangle| = \frac{3}{2} n^2 \epsilon = \frac{3}{2} n^2 \sqrt{1 - l^2/n^2}, \quad (13)$$

which shows that the most eccentric orbits, those for which the angular momentum is lowest, have, as expected, the greatest electric dipole moments.

Now, since $\langle \mathbf{d} \rangle$ is in the direction of \mathbf{A} and $|\mathbf{A}| = \epsilon$, we have

$$\langle \mathbf{d} \rangle = (3/2)n^2 \mathbf{A}. \quad (14)$$

The potential energy of the dipole in an electric field is given by $-\mathbf{p} \cdot \mathbf{F}$ so that the energy shift due to an electric field in the z direction is

$$\Delta E = -(3/2)n^2 A_z F, \quad (15)$$

where the magnitude of the electric field must be low enough to avoid significant distortion of the shape of the Keplerian orbit. This condition can be met when energy shifts due to quadratic and higher powers of F are negligible.

The energy change in Eq. (15) is the classical first-order Stark shift on a bound electron subject to a $1/r$ potential. This may be compared to the quantum mechanical result by inserting the quantized values of A_z , Eq. (4), which gives

$$\Delta E = -(3/2)n(n_1 - n_2)F. \quad (16)$$

This is identical to the quantum mechanical result. This expression, which is sometimes referred to as the Pauli sub-

stitution,¹¹ shows the correspondence between the classical dipole moment, for which the n 's are not quantized, and the quantum mechanical dipole moment. Note that the maximum and minimum Stark shifts are the same in the classical and quantum mechanical cases.

It is well known that central potentials lead to conservation of total angular momentum, which in turn restricts the motion to a plane. Thus for a pure Coulomb or gravitational potential the closed elliptical orbit remains in a plane perpendicular to the angular momentum vector. For example, the earth's motion is confined to the plane of the ecliptic. Since application of a Stark field to a hydrogen atom destroys the central nature of the potential, the classical motion of the orbiting electron will no longer be restricted to a plane. If however, the field is weak, as we have presumed in this paper, then the orbit will resemble an ellipse, the plane and shape of which do not remain constant. Under these circumstances the electron moves around the orbit in a time short compared with the time required for changes in the plane and shape of the trajectory.² Thus the time dependencies of the shape and orientation of the orbit provide a classical picture of the hydrogen atom subjected to a weak constant electric field.

In order to determine the motion of the orbit we examine the time dependence of the Lenz and total angular momentum vectors using Hamilton's equations:

$$\dot{\mathbf{p}} = -\nabla H, \quad \text{and} \quad \dot{\mathbf{r}} = \nabla_{\mathbf{p}} H, \quad (17)$$

where ∇ and $\nabla_{\mathbf{p}}$ are the del operators for space and momentum differentiation, respectively. The Hamiltonian may be written

$$H = H_{\text{Coulomb}} + V(\mathbf{r}), \quad (18)$$

where H_{Coulomb} is the Hamiltonian for a pure Coulomb potential. Taking the gradient of Eq. (18), we have

$$\dot{\mathbf{p}} = \dot{\mathbf{p}}_c - \nabla V(\mathbf{r}), \quad (19)$$

where we use the subscript c to denote quantities for a pure Coulomb potential. We have then

$$\begin{aligned} \dot{\mathbf{L}} &= \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \\ &= \dot{\mathbf{L}}_c - \mathbf{r} \times \nabla V(\mathbf{r}) \end{aligned}$$

so that

$$\dot{\mathbf{L}} = -\mathbf{r} \times \nabla V(\mathbf{r}) \quad (20)$$

and, since $\nabla V = \mathbf{F}$

$$\dot{\mathbf{L}} = -\mathbf{r} \times \mathbf{F}. \quad (21)$$

Since the electron motion is rapid compared to changes in the orbit, we may replace \mathbf{r} by $\langle \mathbf{r} \rangle = -\langle d \rangle$, which has already been evaluated, Eq. (14). Thus

$$\dot{\mathbf{L}} = \left(\frac{3}{2}\right)n^2(\mathbf{A} \times \mathbf{F}). \quad (22)$$

Note, that, since $(3/2)n^2\mathbf{A}$ is the electric dipole moment, and \mathbf{L} is simply the torque on the dipole, we arrive at the familiar expression for the torque on a dipole $\mathbf{d} \times \mathbf{F}$.

It should be borne in mind that the appearance of n in Eq. (22) in no way signifies that this is a quantum mechanical result. In fact, from Kepler's third law the semimajor axis is given by $a = n^2$ (see Table I), so this equation is indeed a purely classical one. We note that, since \mathbf{F} is taken to be in the z direction, $\dot{L}_z = 0$, so that, even with the applied field the z component of angular momentum is conserved. This is of course consistent with the use of L_z as one of the commuting operators in the parabolic coordinate treatment of the Stark effect.

The fact that A_z commutes with both H and L_z in the presence of the applied field must be based on conservation of A_z in the classical problem. Indeed, we find an equation for $\dot{\mathbf{A}}$ that is analogous to that for $\dot{\mathbf{L}}$. The derivation is slightly more involved so we outline the steps. Differentiating the defining equation for \mathbf{A} , Eq. (1), using Eq. (19) and regrouping, we have

$$\begin{aligned} \dot{\mathbf{A}} &= \dot{\mathbf{p}} \times \mathbf{L} + \mathbf{p} \times \dot{\mathbf{L}} - \dot{\mathbf{r}} \\ &= [\dot{\mathbf{p}}_c - \nabla V(\mathbf{r})] \times \mathbf{L} + \mathbf{p} \times \dot{\mathbf{L}} - \dot{\mathbf{r}} \\ &= (\dot{\mathbf{p}}_c \times \mathbf{L} - \dot{\mathbf{r}}) - \mathbf{F} \times \mathbf{L} + \mathbf{p} \times \dot{\mathbf{L}}. \end{aligned} \quad (23)$$

Now, the term in brackets in Eq. (23) is $\dot{\mathbf{A}}_c$, the time derivative of the Lenz vector for a pure Coulomb potential but $\dot{\mathbf{A}}_c = 0$ for the $1/r$ potential so that

$$\dot{\mathbf{A}} = -\mathbf{F} \times \mathbf{L} + \mathbf{p} \times \dot{\mathbf{L}}. \quad (24)$$

We must not at this point use Eq. (22) for $\dot{\mathbf{L}}$ since \mathbf{r} has not yet been replaced by $\langle \mathbf{r} \rangle$. Using Eq. (21) we have

$$\begin{aligned} \mathbf{p} \times \dot{\mathbf{L}} &= \mathbf{p} \times (-\mathbf{r} \times \mathbf{F}) \\ &= \mathbf{F}(\mathbf{p} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{p} \cdot \mathbf{F}). \end{aligned} \quad (25)$$

Now, in order to take the time average we must evaluate $\langle \mathbf{p} \cdot \mathbf{r} \rangle$ and $\langle \mathbf{r}(\mathbf{p} \cdot \mathbf{F}) \rangle$. The first is zero since

$$\mathbf{p} \cdot \mathbf{r} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = \frac{1}{2} \frac{d}{dt} r^2 \quad (26)$$

and the average over a period of an exact differential of a periodic function vanishes.

In order to evaluate $\langle \mathbf{r}(\mathbf{p} \cdot \mathbf{F}) \rangle$ it is, as usual, helpful to know the answer beforehand. Consider the product $\mathbf{F} \times \mathbf{L}$, which may be written

$$\mathbf{F} \times \mathbf{L} = \mathbf{F} \times (\mathbf{r} \times \mathbf{p}) = \mathbf{r}(\mathbf{F} \cdot \mathbf{p}) - \mathbf{p}(\mathbf{F} \cdot \mathbf{r}). \quad (27)$$

But,

$$\frac{d}{dt} [\mathbf{r}(\mathbf{r} \cdot \mathbf{F})] = \mathbf{r}(\mathbf{p} \cdot \mathbf{F}) + \mathbf{p}(\mathbf{r} \cdot \mathbf{F}) \quad (28)$$

is an exact differential which averages to zero. Therefore, $\langle \mathbf{r}(\mathbf{p} \cdot \mathbf{F}) \rangle = -\langle \mathbf{p}(\mathbf{r} \cdot \mathbf{F}) \rangle$, and

$$\dot{\mathbf{A}} = \left(\frac{3}{2}\right)(\mathbf{L} \times \mathbf{F}). \quad (29)$$

For an applied electric field in the z direction $\dot{A}_z = 0$ as discussed above.

The coupled symmetric equations (22) and (29) for $\dot{\mathbf{A}}$ and $\dot{\mathbf{L}}$ can be manipulated to show that the perpendicular vectors \mathbf{A} and \mathbf{L} precess about the electric field vector. Furthermore, the changing magnitude of \mathbf{A} shows that the shape of the elliptical orbit is changing. The equations are easily uncoupled by differentiating one and substituting from the other. For example, differentiation of $\dot{\mathbf{A}}$ and substitution for $\dot{\mathbf{L}}$ yields

$$\begin{aligned} \ddot{\mathbf{A}} &= -\left(\frac{3}{2}\right)n^2[\mathbf{F} \times (\mathbf{A} \times \mathbf{F})] \\ &= -\left(\frac{3}{2}\right)n^2[\mathbf{A}(\mathbf{F} \cdot \mathbf{F}) - \mathbf{F}(\mathbf{F} \cdot \mathbf{A})]. \end{aligned} \quad (30)$$

Since $\mathbf{F} = F\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit vector in the z direction, $\dot{A}_z = 0$, as it must. The other components of $\dot{\mathbf{A}}$ have the form

$$\dot{A}_i(t) = -\left(\frac{3}{2}\right)nF^2 A_i(t) \quad (31)$$

leading to

$$A_i(t) = C_1 \cos(\omega_s t) + C_2 \sin(\omega_s t), \quad (32)$$

where C_1 and C_2 are constant and $\omega_s = (3/2)nF$ is called the Stark frequency. Taking $A_y(t=0) = 0$ and the amplitudes as A_{x0} and A_{y0} , we have

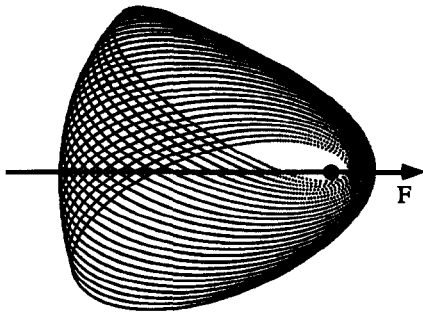


Fig. 2. Trajectory of the electron in a classical hydrogen atom ($n = 11$, $m = 1$, $k = 4$) that is subjected to an external electric field F . The trajectory was generated by numerical solution of Hamilton's equations of motion with $|\mathbf{F}| = 2.917 \times 10^{-7}$ a.u. (1500 V/cm). The nearly Keplerian orbits of the electron are slightly distorted and precess around the electric field vector as discussed in the text.

$$A_x(t)^2 + A_y(t)^2 = A_{x0}^2 \cos^2(\omega_s t) + A_{y0}^2 \sin^2(\omega_s t), \quad (33)$$

which, together with $A_z(t) = \text{const}$, shows that the Lenz vector traces an ellipse in a plane perpendicular to the direction of the applied electric field. Further, the frequency of the motion is ω_s . The changing shape of the ellipse is due to variations in the length of the minor axis while the length of the major axis, and thus the energy, remains constant. Figure 2 shows a series of electronic orbits that illustrate the picture described above.

Similar treatment of the angular momentum leads to analogous equations for the motion of L . It too outlines an ellipse with frequency ω_s in a plane perpendicular to the z axis. The major axes of the two ellipses are perpendicular. As required by the general condition $\mathbf{A} \cdot \mathbf{L} = 0$, both vectors precess about the z axis in the same direction. We have the following picture: The plane of the elliptical orbit of the electron precesses about the electric field vector with frequency ω_s , while, simultaneously, the shape of the ellipse pulsates, also with frequency ω_s .

It is not an accident that the frequency of the classical

motion of A and L is precisely equal to the energy difference ΔE between adjacent quantum mechanical Stark states divided by \hbar [see Eq. (7)]. For a given value of n , different Stark energies correspond to different orientations of the z component of the permanent electric dipole moment which, as we have seen, are quantized. Thus \hbar times the frequency of motion of the orbit must be the same as the separation between adjacent Stark states because, classically, radiation between states must occur at the frequency of the electronic motion, ω_s , in this case, as required by Bohr's correspondence principle.

ACKNOWLEDGMENTS

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⁵See for example, S. Borowitz, *Fundamentals of Quantum Mechanics* (W. A. Benjamin, New York, 1967), pp. 304–307.

⁶See, for example, H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1981, second edition), pp. 102–105.

⁷This vector has been referred to by a variety of names and combinations of names. For a discussion the designation of A , see Ref. 6, footnote on p. 103. For brevity we have chosen to refer to A as the Lenz vector.

⁸For a graphical representation of the electron probability density, see D. Kleppner, M. G. Littman, and M. L. Zimmerman, in *Rydberg States of Atoms and Molecules*, edited by R. F. Stebbings and F. B. Dunning (Cambridge U. P., Cambridge, 1983), pp. 79–80.

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¹¹A. R. P. Rau, "Rydberg States in Crossed Fields: The Gyropendulum," *Phys. Rev. A* **34**, 4501–4503 (1986).

ROBERT R. WILSON: THE VALUE OF FERMILAB

"Senator Pastore, I and my colleagues will be spending a good part of our lives building and using this machine. We have a deep and very personal commitment to it." [Wilson] launched into an impassioned argument for the accelerator, and Senator Pastore was being impassioned right back. Among other things the senator said: "Is there anything connected [with] this accelerator that in any way involves the security of the country?" Wilson responded: "No, sir; I do not believe so." Senator Pastore persisted: "Nothing at all?" Finally Wilson said: "[The accelerator] has to do with: Are we good painters, good sculptors, great poets?... It has nothing to do with defending our country, *except to make it worth defending.*"

Dennis Flanagan, *Flanagan's Version* (Vintage Books, New York, 1988), p. 16.