Contents

Calculate the total force on a charge Q located at the center or a regular polygon having $2N + 1$ sides, N an integer, with a point charge q located on each corner.

Solution:

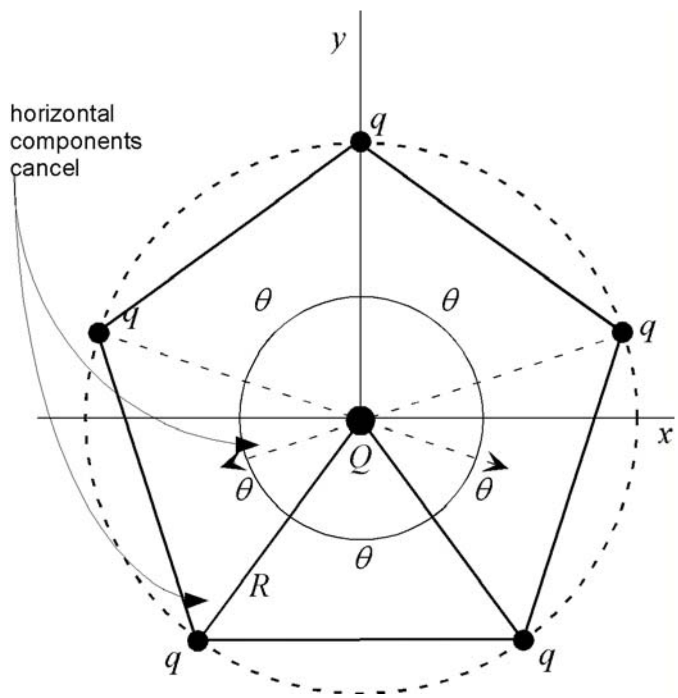

FIGURE 1. The geometry of the problem using a pentagon, that is, $N = 2$.

The circle represents the circle of radius R within which the polygon is inscribed. The angle between successive charges on the circumference is $\theta = 2\pi/(2N+1)$.

Place one charge on the top, the y-axis as shown. There are then N charges symmetrically located on each side of the y-axis. The angle between successive force vectorIt is clear that the horizontal components cancel each other so there is no horizontal component of force.

Let $F = \dfrac{1}{4\pi\epsilon_0} \cdot \dfrac{qQ}{R^2}$ where $R =$ the radius of the circle in which the polygon is inscribed. The total force F_T is in the $-y$ direction and is given by

$$\begin{aligned} F_T &= -F - 2F\cos\theta - 2F\cos(2\theta) - 2F\cos(3\theta)\ldots \\ &= -F\left[1 + 2\sum_{n=1}^{N}\cos(n\theta)\right] \end{aligned} \tag{1}$$

Now evaluate the summation

$$\begin{aligned} \sum_{n=1}^{N}\cos(n\theta) &= \sum_{n=0}^{N}\cos(n\theta) - 1 \\ &= -1 + \mathrm{Re}\left[\sum_{n=0}^{N} e^{in\theta}\right] \end{aligned} \tag{2}$$

The summation for which we require the real part is actually a finite geometric series with common ratio $r = e^{i\theta}$ so, as a bonus, we derive the sum of such a series. Let the sum of the series S_N be given by

$$S_N = a + ar^1 + ar^2 + ar^3 + \cdots ar^N \quad \text{Note that there are } N+1 \text{ terms} \tag{3}$$

Now multiply the series through by r to obtain

$$rS_N = ar^1 + ar^2 + ar^3 + ar^4 + \cdots ar^{N+1} \quad \text{Note that there are still } N+1 \text{ terms} \tag{4}$$

Now subtract Equation 3 from Equation 4 to obtain

$$S_N - rS_N = a - ar^{N+1} \qquad \text{or} \qquad S_N = a\left(\frac{1 - r^{N+1}}{1 - r}\right) \tag{5}$$

Then, for the series we require in Equation 2 we have

$$\begin{aligned} \mathrm{Re}\left[\sum_{n=0}^{N} e^{in\theta}\right] &= \mathrm{Re}\left[\frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}}\right] \\ &= \mathrm{Re}\left[\frac{e^{i(N+1)\theta/2}}{e^{i\theta/2}} \cdot \frac{e^{i(N+1)\theta/2} - e^{-i(N+1)\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right] \\ &= \frac{\sin\left(\frac{N\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\mathrm{Re}\left[e^{iN\theta/2}\right] \\ &= \frac{\sin\left[\frac{(N+1)\theta}{2}\right]\cos\left(N\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \end{aligned} \tag{6}$$

To simplify the numerator use

$$\sin A \cos B = \frac{1}{2}\left[\sin\left(A-B\right)+\sin\left(A+B\right)\right] \tag{7}$$

and substitute $\theta = 2\pi/\left(2N+1\right)$, the condition that puts the charges on a regular polygon having an odd number of sides

$$\sin\left[\frac{(N+1)}{(2N+1)}\pi\right]\cos\left[\frac{N}{(2N+1)}\pi\right] = \frac{1}{2}\left\{\sin\left[\frac{\pi}{(2N+1)}\right]+\sin\left[\frac{(2N+1)}{(2N+1)}\pi\right]\right\} \tag{8}$$

$$= \frac{1}{2}\left\{\sin\left[\frac{\pi}{(2N+1)}\right]+\sin\left[\frac{(2N-1)\,\pi}{(2N+1)}\right]\right\}$$

$$= \frac{1}{2}\sin\left[\frac{\pi}{(2N+1)}\right] \tag{9}$$

Then

$$\sum_{n=0}^{N}\cos\left(n\theta\right) = \frac{\frac{1}{2}\sin\left[\frac{\pi}{(2N+1)}\right]}{\sin\left[\frac{\pi}{(2N+1)}\right]} = \frac{1}{2} \tag{10}$$

Using Equation 2 we have

$$\sum_{n=1}^{N}\cos\left(n\theta\right) = \frac{1}{2} - 1 = -\frac{1}{2} \tag{11}$$

Finally then,

$$\begin{aligned} F_T &= -F\left[1+2\sum_{n=1}^{N}\cos\left(n\theta\right)\right] \\ &= -F\left[1+2\left(-\frac{1}{2}\right)\right] \\ &= 0 \end{aligned} \tag{12}$$

and the net force is zero.