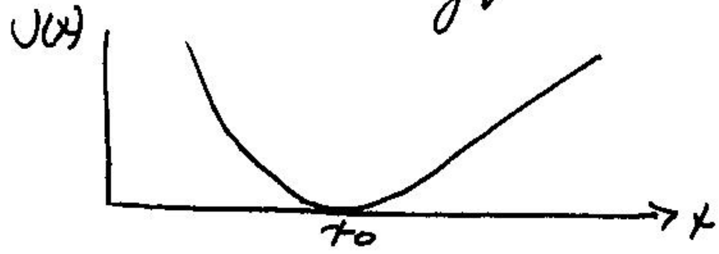
Chapter 3

Assume an arbitrary potential energy function $U(x)$.



For simplicity choose $U(x = x_0) = 0$

Expand $U(x)$ in a Taylor's series about x_0

$$U(x) = \cancel{U(x)\Big|_{x=x_0}}^{\,0} + \cancel{\frac{dU(x)}{dx}}^{\,0}\Big|_{x=x_0}(x-x_0)$$

$$+ \frac{d^2U(x)}{dx^2}\Big|_{x=x_0}\frac{(x-x_0)^2}{2!} + \cdots$$

$\therefore$ The leading term in the expansion is

$$U(x) = C\,\frac{(x-x_0)^2}{2}$$

Now, if we assume a restoring force proportional to the displacement, i.e. $F(x) = -kx$ and translate the x-axis so that $x_0 = 0$ we have $\vec{F} = -\nabla U \Rightarrow U = \frac{1}{2}kx^2$

We see that a Hooke's law force (proportional to the displacement) is the first approximation to any potential well.

This, of course, results in a simple harmonic oscillator (SHO).

$$F = m\ddot{x} = -kx \implies \ddot{x} + \omega_0^2 x = 0$$

where $\omega_0^2 = k/m$

To solve this equation we try a solution of the form $x = e^{mt}$ which leads to

$$m^2 + \omega_0^2 = 0 \implies m = \pm i\omega_0$$

so
$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$
$$= A(\cos\omega_0 t + i\sin\omega_0 t) + B(\cos\omega_0 t - i\sin\omega_0 t)$$
$$= (A+iB)\cos\omega_0 t + (A-iB)\sin\omega_0 t$$
$$= C\cos\omega_0 t + D\sin\omega_0 t$$

Thus, the response, $x(t)$, may be written as a linear combination of $\cos\omega_0 t$ & $\sin\omega_0 t$.

We may also cast it into a form that is convenient in some application.

Let $C = E\sin\delta$ & $D = E\cos\delta$ so that

$$x(t) = E[\sin\delta\cos\omega_0 t - \cos\delta\sin\omega_0 t]$$
$$= E\sin(\omega_0 t - \delta)$$