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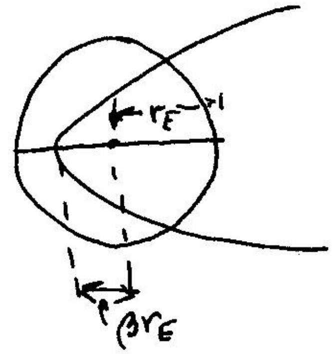


Orbit of parabola $(\epsilon = 1)$:

$$\frac{\alpha}{r} = 1 + \cos\theta$$

Let $r(\theta = 0) = \beta r_E$

$$\therefore \quad \frac{\alpha}{\beta r_E} = 2 \Rightarrow \alpha = 2\beta r_E = \frac{\ell^2}{\mu k}$$

$$TME = 0 = \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

so $$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\cdot\frac{k}{r} - \frac{\ell^2}{\mu^2 r^2}}$$

$\therefore$

$$T = \int dt = 2\int_{\beta r_E}^{r_E} \frac{dr}{\sqrt{\frac{2}{\mu}\left(\frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}}$$

= time spent within orbit

$$= \sqrt{\frac{2\mu}{k}} \int_{\beta r_E}^{r_E} \frac{r\,dr}{\sqrt{r - \beta r_E}}$$

$$= \sqrt{\frac{2\mu}{k}}\left[\frac{2}{3} r_E^{3/2}(2\beta + 1)\sqrt{r - \beta r_E}\right]_{\beta r_E}^{r_E}$$

$$= \sqrt{\frac{2\mu}{k}}\left[\frac{2}{3} r_E^{3/2}(2\beta+1)\sqrt{1-\beta}\right]$$

For the earth (Kepler's 3rd law): $\tau_E^2 = \frac{4\pi^2 \mu_E}{k'} r_E^3$

so $T$ becomes

$$T = \sqrt{\frac{2\mu}{k}}\,\frac{2}{3}\sqrt{\frac{k'}{\mu_E}}\,\frac{\tau_E}{2\pi}(2\beta+1)\sqrt{1-\beta}\,;$$ where $k = GM_s\mu$, $k' = GM_s\mu_E$

so $$T = \frac{1}{3\pi}\sqrt{2(1-\beta)}\,(1+2\beta)\,\tau_E \quad (\text{where } \tau_E = 1\text{ year})$$

$$\beta = r_{mercury}/r_E = 0.387 \Rightarrow T = 76 \text{ days}$$