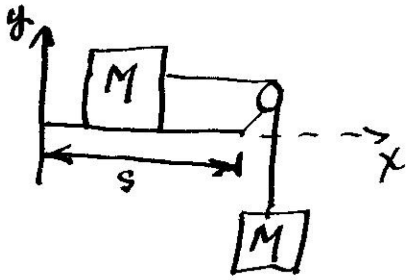
7-10



ℓ = length of string $\therefore (s-x) - y = \ell$

y is negative for this coordinate system.

so $\dot{x} = -\dot{y}$

a) $L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - Mgy = M\dot{y}^2 - Mgy$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 = -Mg - 2M\ddot{y}$$

$\therefore \ddot{y} = -g/2$ from which we get the usual equations.

b) $T_{string} = \frac{1}{2}m\dot{y}^2$; $U_{string} = -\left[\left(\frac{m}{\ell}\right)y\right]g\left(\frac{y}{2}\right)$

mass of hanging string — cm of string

$$\therefore L = M\dot{y}^2 - Mgy + \frac{1}{2}m\dot{y}^2 + \frac{mg}{2\ell}y^2$$

Lagrange eqn: $(2M+m)\ddot{y} - \frac{mg}{\ell}y + Mg = 0$

or $\ddot{y} - \gamma^2 y = Mg$ where $\gamma^2 = mg/[\ell(2M+m)]$

soln to homogeneous eqn: $y_H = Ae^{\gamma t} + Be^{-\gamma t}$

soln to particular eqn: $y_P = \frac{M}{m}\ell$

$$y = y_H + y_P = Ae^{\gamma t} + Be^{-\gamma t} + \frac{M}{m}\ell$$

If $y(t=0) = 0 = \dot{y}(t=0) \Rightarrow A+B = -\frac{M}{m}\ell$

and $A = +B \Rightarrow A = B = -\frac{M}{2m}$

Then
$$y(t) = \frac{M}{m}\ell\left[1 - \left(\frac{e^{\gamma t} + e^{-\gamma t}}{2}\right)\right]$$
$$= \frac{M}{m}\ell\left[1 - \cosh(\gamma t)\right]$$