# Traveler-point dynamics: A 3-vector bridge to curved spacetimes

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Locally-defined “flat-patch” parameters, chosen to be minimally frame-variant, can be useful for describing the perspective of a *single observer* in accelerated frames and in curved spacetimes. In particular the metric-equation’s scalar proper-time, and the 3-vectors proper-velocity and proper-acceleration, are useful because they don’t rely on extended arrays of synchronized clocks, which may be hard to find. This use of an observer’s “proper reference frame” is where diverse metric definitions of extended-simultaneity converge to agree on a set of locally-approximate “geometric” (i.e. connection-coefficient) forces, which in turn further prepare intro-physics students for “flat-patch” engineering in curved spacetimes. Newton’s approximation to gravity is the oldest example of this. The strategy also opens the door to engineering nomograms that allow one to track *arbitrary* local and extended radar time trajectories from the perspective of accelerated and curved-spacetime travelers, as well as real-time single-traveler trajectories that are exact at any speed in flat spacetime, and approximate around gravitating masses.

**CONTENTS**



## I. INTRODUCTION

In an excellent 1968 article[?] , Robert W. Brehme discussed the advantage of teaching relativity with four-vectors. This is not done for intro-physics courses. One reason may be that spacetime’s 4-vector symmetry is broken *locally* into (3+1)D, so that e.g. engineers working in curved spacetimes[?] (including that here on earth) will naturally want to describe time-intervals in seconds and space-intervals e.g. in meters. In this note, we therefore explore in depth an idea about use of minimally frame-variant scalars and 3-vectors, suggested by many but perhaps first hinted at in Percy W. Bridgman’s 1928 ruminations[?] about the promise of defining 3-vector velocity using traveler proper time instead of an ”extended” time system.

The problem with the former, of course, is that minimally frame-variant quantities like proper time, 3-vector proper velocity[?] , and 3-vector proper acceleration[?] are inherently local to one traveler’s perspective, hence we refer to them as ”traveler-point” quantities. The advantage is that they are either frame-invariant (in the case of proper time and the magnitude of proper-acceleration) or synchrony-free[?] (i.e. do not require an extended network of synchronized clocks which is often unavailable in curved space time and accelerated frames).

In a larger context, general-relativity revealed a century ago why we can get by with using Newton’s laws in spacetime so curvy that it’s tough to jump higher than a meter. It’s because those laws work locally in all frames of reference[?] [?] , provided we recognize that motion is generally affected by both proper and “geometric” (i.e. accelerometer-invisible connection-coefficient) forces. The emergent “metric-first” approach to exploring curved spacetimes with calculus[?] can choose to restrict *all measurement* to locally-flat patches of spacetime in this context, which can then be sewn together by a global metric[?] [?] .

This opens the door to engineering use of spatial 3-vectors and temporal scalars at any speed, and locally in curved spacetimes, for engineers who have little interest in ignoring the local (3+1)D break in our global 4-vector symmetry which gives rise to fundamental differences in the way that space and time are experienced. The metric equation also contents itself with a single definition of extended simultaneity (cf. Fig. 1) i.e. that associated with the set of book-keeper (or map) coordinates with which one chooses to describe one’s local patch (for measurement) and one’s global spacetime metric (e.g. for formally defining extended simultaneity). Avoiding the Lorentz transform’s need for two frames with synchronized clocks in special relativity[?] of course opens the door to simple treatments of accelerated motion in flat spacetime as well.

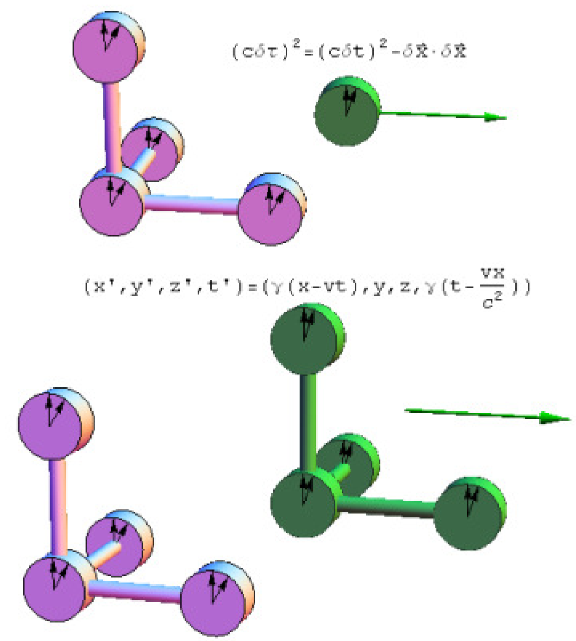


FIG. 1. Even one extended-frame with synchronized clocks may be more than we have in curved spacetime.

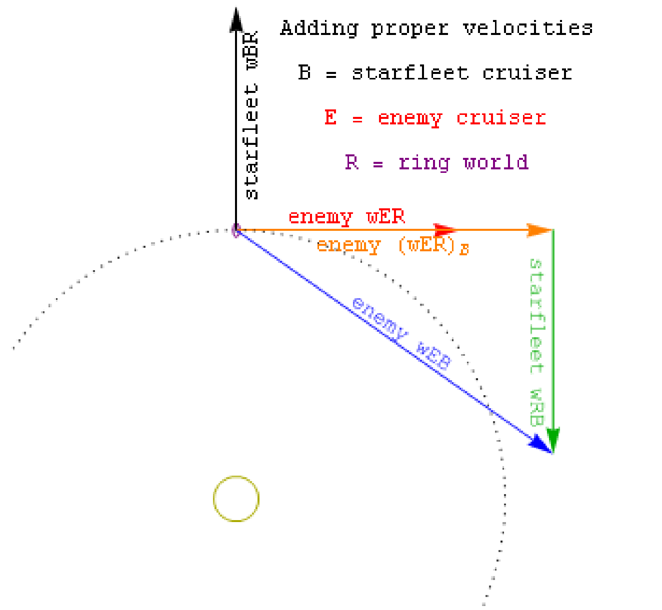


FIG. 2. Proper-velocity addition for a sci-fi puzzler, in which your starfleet battle-cruiser drops in from hyperspace heading away from a nearby star, only to find the enemy ship heading off in another direction.

## II. PROPER TIME

Proper-time is the time elapsed locally along the world-line of any physical clock (broadly defined). It is also the frame-invariant time-interval $\delta\tau$ that shows up in metric equations of the form $(c\delta\tau)^2 = \Sigma_\mu \Sigma_\nu g_{\mu\nu} \delta X_\mu \delta X_\nu$, which in the flat-space Cartesian case takes the Pythagorean form $(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2 - (\delta y)^2 - (\delta z)^2$, where book-keeper or map coordinates are found on the right-hand side of the equation, and $c$ is the "lightspeed" spacetime constant for converting e.g. seconds into meters.

When simultaneity is defined by a flat-spacetime book-keeper coordinate frame, time-elapsed on traveling clocks is always "dilated" (e.g. spread out), since from the metric equation the differential-aging or Lorentz factor $\gamma \equiv dt/d\tau \geq 1$. Expressions for this in terms of velocity will be discussed below.

We won't discuss the utility of proper-time $\tau$ in detail here because it is widely used now even by introductory textbook authors[?] . However it's worth pointing out that early treatments of special relativity were so focused on the equivalence between frames, each with their own extended set of synchronized clocks, that proper-time (if discussed[?] ) was generally a late-comer to the discussion, and of course used only for "rest frame" (i.e. unaccelerated) travelers[?] .

Moving yardsticks also have a "proper-length" associated with them, which can be useful too. However this is a "non-local" quantity not defined by the metric equation, the concept of "rigid body" is only an approximation in spacetime, and length contraction itself involves three separate events as distinct from the two involved in time-dilation. Hence we don't include proper-length in a collection of those traveler-point parameters which show promise of wide-ranging usefulness in accelerated frames and curved spacetime.

## III. PROPER VELOCITY

Proper-velocity[?] $\vec{w} \equiv d\vec{x}/d\tau$ is the rate at which book-keeper or map 3-vector position $\vec{x}$ is changing per unit-time elapsed locally on traveler clocks. Because it is proportional to momentum (i.e. $\vec{p} = m\vec{w}$), unlike coordinate velocity $\vec{v} \equiv d\vec{x}/dt$ it has no upper limit. Also unlike coordinate velocity, its measurement does not depend on map-clock readings (synchronized or not) along that world line. Proper-velocity is also represented by the space-like components of a traveling object's velocity 4-vector.

An interesting thought problem for future engineers (as well as teachers) might be to ask if speed-limit signs, in a "Mr. Tompkins" world[?] with a much slower light-speed spacetime constant, would use coordinate-speed or proper-speed. Four criteria to consider might involve the measure's connection to momentum, kinetic-energy, and to reaction-times (after the "warning photons" arrive) for both the driver, and a pedestrian who is tempted to cross the street.

The time-like component of that 4-vector is $c\gamma$, where $\gamma \equiv dt/d\tau$ is the also-useful (but not synchrony free) Lorentz differential-aging or "speed of map-time" factor mentioned above. In flat spacetime, from the metric equation it is easy to show that $\gamma = w/v = \sqrt{1+(w/c)^2} \geq 1$. This works better at high speeds than the version in terms of coordinate speed because there is no difference in the denominator between numbers very near to 1.

This same proper velocity form also emerges directly for circular orbits around a spherically symmetric mass, as the exact azimutal motion factor in the Schwarzchild time-dilation equation $\gamma \equiv dt/d\tau = \gamma_r\sqrt{1+(w_\phi/c)^2}$, where the gravitational time-dilation factor is $\gamma_r \equiv 1/\sqrt{1-(r_s/r)} \geq 1$ and $r \geq r_s$ is distance from the center of a mass $M$ object whose Schwarzschild radius $r_s = 2GM/c^2$. Global positioning system satellites have to consider both of these factors in compensating for the rate of time's passage in their orbits around earth.

For unidirectional velocity addition, the advantage of a collider over an accelerator is easily seen because $w_{AC} = \gamma_{AB}\gamma_{BC}(v_{AB}+v_{BC})$. In other words, the Lorentz-factors multiply even though the coordinate-velocites (which add) never exceed $c$. Proper velocity's direct connection to momentum makes it more relevant to speed-limits e.g. in worlds with a reduced value for lightspeed, and its direct connection to traveler time makes it more relevant to passengers and crew when planning long high-speed trips.

More generally, unlike coordinate-velocities, proper velocities add vectorially provided that one rescales (in magnitude only) the "out of frame" component. In other words $\vec{w}_{AC} = \gamma_{AC}\vec{v}_{AC} = (\vec{w}_{AB})_C + \vec{w}_{BC}$. Here C's view of the out-of-frame proper-velocity $(\vec{w}_{AB})_C$ is in the same direction as $\vec{w}_{AB}$ but rescaled in magnitude by a factor of $(\gamma_{BC} + (\gamma_{AB}-1)\vec{w}_{\rm BC}\cdot\vec{w}_{\rm AB}/w^2_{\rm AB}) \geq 0$. Hence vector diagrams, and even the original low-speed equation for velocity-addition (almost), survive intact when proper-velocity is used.

This would open the door to new settings for those relative motion problems (that students dislike so much). For example, suppose that a starfleet battle-cruiser drops out of hyperspace in the orbital plane of a ringworld, traveling at a proper-velocity of 1[ lightyear/traveler-year] radially away from the ringworld's star. An enemy cruiser drops out of hyperspace nearby at the same time, traveling 1[ly/ty] in the rotation-direction of the ringworld's orbit, and in a direction perpendicular to the starfleet cruiser's radial-trajectory. What is the proper-velocity (magnitude in [ly/ty] and direction) of the enemy cruiser relative to the starfleet ship? The vector solution to this is illustrated in the Fig. 2.

## IV. PROPER ACCELERATION

The felt or proper-acceleration 3-vector $\vec{\alpha}$ is defined as the acceleration felt by accelerometers (broadly defined) that are moving with an object[?] relative to the tangent free-float (or geodesic as a generalization of "inertial") frame. In flat spacetime its magnitude is the frame-invariant magnitude of the net-acceleration 4-vector $A^\lambda \equiv \frac{\delta U^\lambda}{\delta\tau} \equiv \frac{\delta^2 X^\lambda}{\delta\tau^2}$. This four-vector has a null time-component in the frame of the accelerated traveler, from whose "traveler-point" perspective its 3-vector direction is defined.

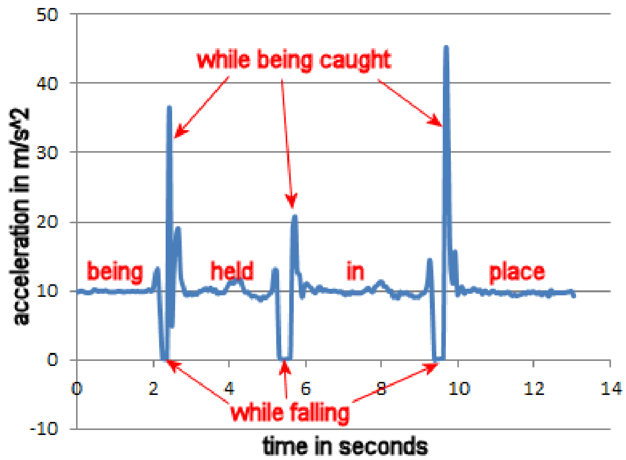


FIG. 3. Accelerometer data from a phone dropped and caught three times. During the free-fall segments, the accelerometer reading drops to zero because gravity is a geometric force (like centrifugal) which acts on every ounce of the phone's structure, and is hence not detected. The positive spikes occur when the fall is arrested as the falling phone is caught (also by hand) before it hit the floor.

In flat spacetime, this 3-vector is simply related to the rate of proper-velocity change through $\vec{\alpha} = \frac{\delta\vec{w}}{\delta t}_{||\vec{w}} + \gamma\frac{\delta\vec{w}}{\delta t}_{\perp\vec{w}}$, which in the unidirectional case reduces to $\vec{\alpha} = \frac{\delta\vec{w}}{\delta t}$. This link between proper acceleration and proper velocity will be important when we look at traveler-point dynamics in the next section.

To illustrate what happens in curved spacetime (and accelerated frames), try running an accelerometer app on your phone and performing a hold, drop, catch sequence as shown in Fig. 3. Because accelerometers only detect proper-accelerations, like that due to the upward force of your hand when holding the phone in place before the drop, nothing is detected when the phone is released even through the net acceleration 4-vector in that case has a vertically downward component of $g$.

Thus in curved spacetimes and accelerated frames, understanding motion requires a recognition of geometric (connection-coefficient) effects which thankfully (following Newton) can often be approximated locally as gravitational and inertial forces. The bottom line is that in flat spacetime motion is explained by the forces that cause proper acceleration, but as soon as we move to curved spacetimes and accelerated frames (even here on earth) a second kind of cause comes into play. As we'll see, these geometric effects act on every ounce of their target, vanish locally from the perspective of a comoving free-float frame, and are therefore undetected by on-board accelerometers.

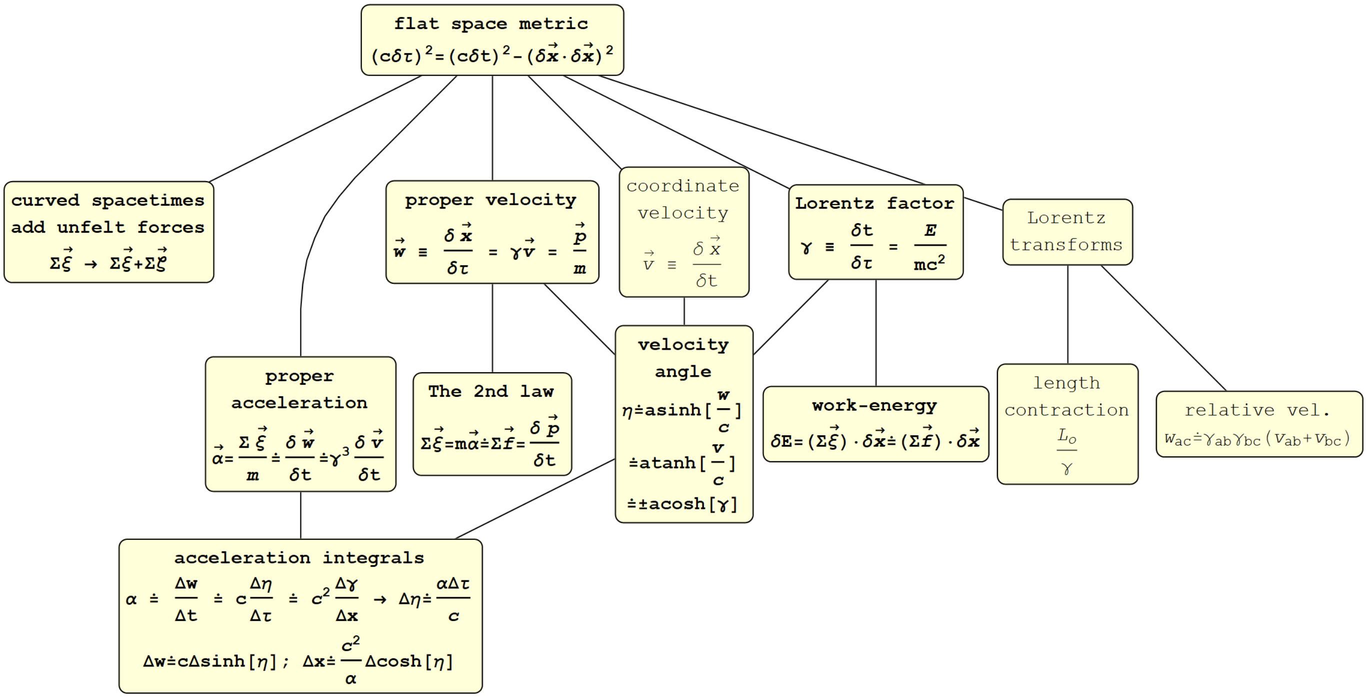


FIG. 4. Metric-1st equations of (1+1)D motion. The dot over the equals sign means this works only for unidirectional motion. The Lorentz transform section at far right is the only place which requires more than one map (or bookkeeper) reference frame of yardsticks and synchronized clocks.

Returning to flat spacetime for the moment, let's also consider the equations of constant proper-acceleration (cf. Fig. 4). In flat spacetime, it is proper and not coordinate acceleration that can be (and normally is) held constant.

If we stick to (1+1)D i.e. unidirectional motion, constant proper acceleration yields some wonderfully simple integrals of motion, namely $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x$, where "hyperbolic velocity angle" or rapidity $\eta = \sinh^{-1}[w/c] = \tanh^{-1}[v/c]$. These are approximated by the familiar introl-physics relationships $a = \Delta v/\Delta t = \frac{1}{2}\Delta(v^2)/\Delta x$ at low speed, but also allow beginning students to explore interstellar constant proper-acceleration round-trip problems almost as easily as one does low-speed constant acceleration problems.

In (3+1)D the integrals are also analytical, but messy and perhaps best expressed[?] in terms of a characteristic time $\tau_o$ equal to $\sqrt{(\gamma_o+1)/2}c/\alpha$, which connects to a hyperbolic velocity angle $\tau/\tau_o$. Here $\gamma_o$ is the "aging factor at turnaround", which for unidirectional motion yields the (1+1)D rapidity discussed above.

## V. CAUSES OF MOTION

We've discussed a robust strategy for describing motion (kinematics) at any speed plus locally in curved spacetime & accelerated frames, using (3+1)D parameters which are either frame-invariant or synchrony free. The causes of motion (dynamics) on the other hand are traditionally described by a connection between net 3-vector force as the cause, and the second time derivative of position where e.g. at low speed we normally write $\Sigma\vec{F} = m\delta^2\vec{x}/\delta t^2$.

To extend this (3+1)-vector analysis to high speeds and curved spacetimes we simply follow the tradition already established with gravity, but add some new notation. For instance, we might denote the *proper forces* (with frame-invariant magnitudes which are felt by on-board accelerometers) with the greek letter xi ($\vec{\xi}$). For inertial frames in flat spacetime, then, $\Sigma\vec{\xi} \equiv m\vec{\alpha} = m\delta^2\vec{x}/\delta\tau^2$.

In the "local patch" of curved spacetimes and accelerated frames, we can often approximate the effects of metric connection coefficients on the equation of motion by imagining geometric forces which: (a) are unfelt by on-board accelerometers, (b) act on every ounce of an object's being, and (c) disappear when seen from the vantage point of a local "free-float frame". We denote these *geometric forces* (like gravity and inertial forces such as centrifugal) with the greek letter zeta ($\vec{\zeta}$). Locally then we can write $\Sigma\vec{\xi} + \Sigma\vec{\zeta} \simeq m\delta^2\vec{x}/\delta\tau^2$.

Although these force expressions work well for tracking map position versus time, energies, accelerometer readings and differential aging, etc., in the special case of high speed non-unidirectional motion a kinematic correction is needed when looking at momentum transfers (e.g. in collisions). This is because rates of proper velocity change (proportional to momentum change) are frame variant

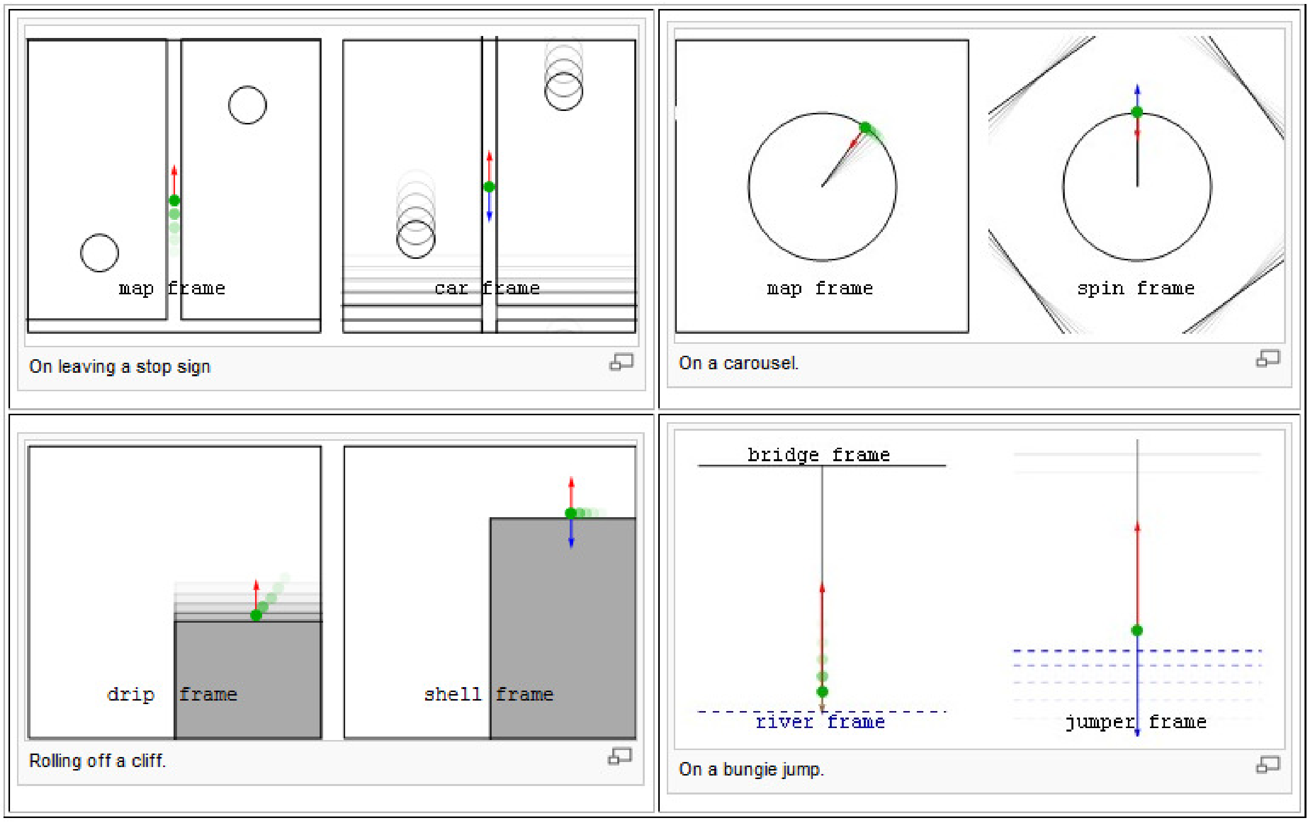


FIG. 5. Two views of proper (red) and geometric (dark blue or brown) forces in some everyday settings.

even in flat spacetime when proper acceleration is not. As a result $\delta\vec{p}/\delta t$ is equal instead to a net sum of *apparent forces* $\Sigma\vec{f} = \delta\vec{p}/\delta t = m\delta\vec{w}/\delta t$, where $\vec{f} = \vec{\xi}_{||\vec{w}} + (1/\gamma)\vec{\xi}_{\perp\vec{w}}$, depends on that object's proper velocity $\vec{w}$.

The idea that rates of momentum change depend on one's frame of reference may seem odd, but rates of energy change ($\delta E/\delta t = \Sigma\vec{\xi}\cdot\vec{v} \rightarrow \Sigma\vec{f}\cdot\vec{v}$) are frame-variant even at low speeds. This in spite of the fact that when I asked an AAPT audience in 1998 for a show of hands by those who thought that classical rates of an object's energy change are frame-variant, the only hand to go up in the audience was that of Edwin F. Taylor (that year's Oersted Medal winner).

As mentioned above, in curved spacetimes the move from 4-vector to local (3+1)D dynamics is normally done by the "geometric force" approximation, in which motion is seen as caused by adding geometric to proper forces. These relations are inspired by the general-relativistic 4-vector equation of motion, written in force-units as $mDU^\lambda/d\tau - m\Gamma^\lambda_{\mu\nu}U^\mu U^\nu = mdU^\lambda/d\tau$, which may be described as "net proper 4-vector force" + "net geometric 4-vector force" = mass times the "net 4-vector acceleration". As usual here $m$ is frame-invariant mass, upper case $D$ denotes the covariant derivative of 4-velocity component $U^\lambda$, $\Gamma^\lambda_{\mu\nu}$ denotes one of $4\times4\times4 = 64$ connection coefficients defined in terms of metric tensor derivatives, and repeated indices in products like $\Gamma^\lambda_{\mu\nu}U^\mu U^\nu$ are implicitly summed over all (spatial and temporal) values of those indices.

The traveler-point approach expresses this in terms of a locally-useful set of 3-vector and scalar relations, like $\Sigma\vec{\xi} + \Sigma\vec{\zeta} \simeq m\delta^2\vec{x}/\delta\tau^2$ and $\delta E \simeq (\Sigma\vec{\xi} + \Sigma\vec{\zeta})\cdot\delta\vec{x}$. The scalar part connects to energy conservation, while the vector part connects to momentum conservation *locally* through the flat spacetime connection to frame-variant forces $\vec{f}$ discussed above. Both the "geometric force" approximation which gives rise to the $\vec{\zeta}$' vectors, and the connection of "felt plus geometric" forces to rates of momentum change, rely on the general assertions: (a) that spacetime is "locally flat" even in curved spacetimes and accelerated frames, and (b) that the proper acceleration 4-vector is purely spacelike (i.e. a.spatial 3-vector) from the perspective of the traveling object. The locality requirement, of course, may be more severe as spacetime curvature increases.

For example, if we imagine a stationary charge $q$ held in place by some combination of electromagnetic forces in a Schwarzschild (non-spinning spherical mass) potential, the net proper force is the Lorentz 4-vector $qF^\lambda_\beta U^\beta = q\gamma\{\vec{E}\cdot\vec{v}/c, \vec{E}+\vec{v}\times\vec{B}\}$. In traveler-point form[?] for charge $q$ this becomes $\Sigma\vec{\xi} = q\vec{E}' = q(\vec{E}_{||\vec{w}} + \gamma\vec{E}_{\perp\vec{w}} + \vec{w}\times\vec{B})$, where $\vec{E}$ and $\vec{B}$ are the bookkeeper-frame field vectors, and $\vec{E}'$ is only the electric field seen by charge $q$.

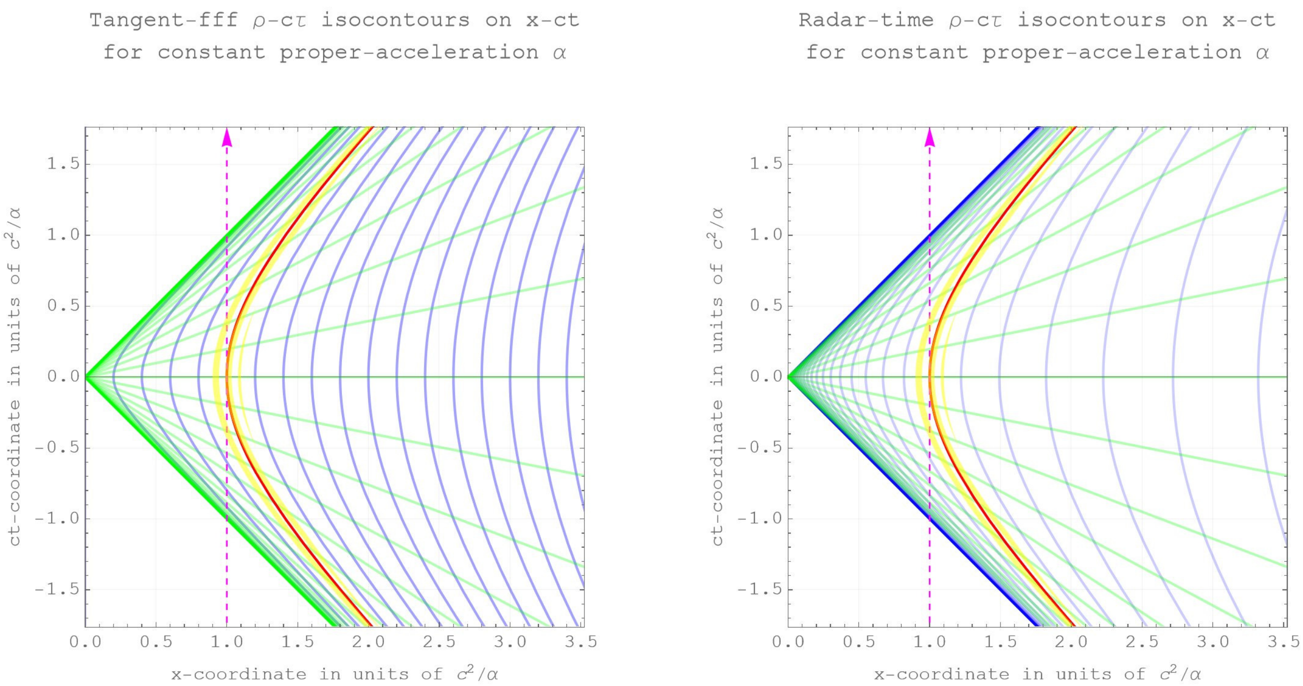


FIG. 6. This compares tangent free-float-frame and radar-time extended simultaneity models for constant proper acceleration. The dashed green line corresponds to the world line of a stationary object, moving forward as if it had been dropped by our accelerated observer (red line) at the turnaround point.

The magnetic field $\vec{B}'$ seen by the charge has no effect on its motion, so that in traveler-point terms such problems become purely electrostatic even in curved spacetime! As an aside in this context, the above relations may be used to show how frame-variant proper forces can in general be decomposed into "static" and "kinetic" components, $\vec{f}_s = \gamma^3 m\vec{a} = \vec{\xi}_{||\vec{w}} + \gamma\vec{\xi}_{\perp\vec{w}}$ and $\vec{f}_k = (\frac{1}{\gamma} - \gamma)\vec{\xi}_{\perp\vec{w}}$, which are analog to the electrostatic and magnetic forces (respectively) associated with electromagnetic fields.

The net geometric force, on the other hand, is obtained by summing the 13 out of 64 non-zero connection coefficients (8 of 40 with independent values) for a Schwarzschild object of mass $M$ and "far-coordinate" radius $r$, to get $\Sigma\vec{\zeta} = -GMm\hat{r}/r^2$. If we wish to support the object in place so that net acceleration 3-vector $\vec{\alpha}$ is zero, then we require an E-field directed in the radial direction for which upward proper force $qE' = GMm/r^2$. One nice thing about discussing this force-balance in traveler-point terms is that all observers, those using far-time in a Schwarzschild potential as well as e.g. those passing by in a star ship accelerating at 1 gee, will be talking about the same thing in terms of time-intervals and forces experienced by the traveler, albeit with position coordinates of their own choosing.

The net proper-force that our cell-phones measure is frame-invariant in magnitude as is proper time, with a 3-vector direction presumably defined in traveler terms. This makes it useful for comparing electromagnetic forces in different frames (e.g. to see how magnetic attraction becomes purely electrostatic), and makes curved spacetime (like that we experience on earth's surface as illustrated in Fig. 5) a bit simpler to understand than we might have imagined in a world where time passes differently according to your location as well as your rate of travel.

Thus Einstein's general relativity, far from invalidating Newton, revealed that the classical laws work *locally* in all frames (including accelerated-frames in curved-spacetime) provided that, in addition to proper-forces, we recognize geometric (i.e. connection-coefficient) forces like gravity and inertial-forces (acceleration "gees", centrifugal, etc.) that "act on every ounce" of an object's being.

An effective potential emerges in accelerated frames and curved spacetimes, because the differential-aging factor $\gamma \equiv dt/d\tau$ (equal to $\sqrt{1/g_{00}}$ for motionless objects) becomes space-dependent. Simple examples of "potential well" depths include:

$$W_{\text{depth}} = \left(\frac{\Delta t_{\text{axis}}}{\Delta t_{\text{off-axis}}} - 1\right) mc^2 \simeq \frac{1}{2} m\omega^2 r^2, \qquad (1)$$

in rotating habitats if we ignore azimuthal forces as radius changes,

$$W_{\alpha L} \simeq \left(\frac{\Delta t_{\text{leading}}}{\Delta t_{\text{trailing}}} - 1\right) mc^2 \simeq m\alpha L, \qquad (2)$$

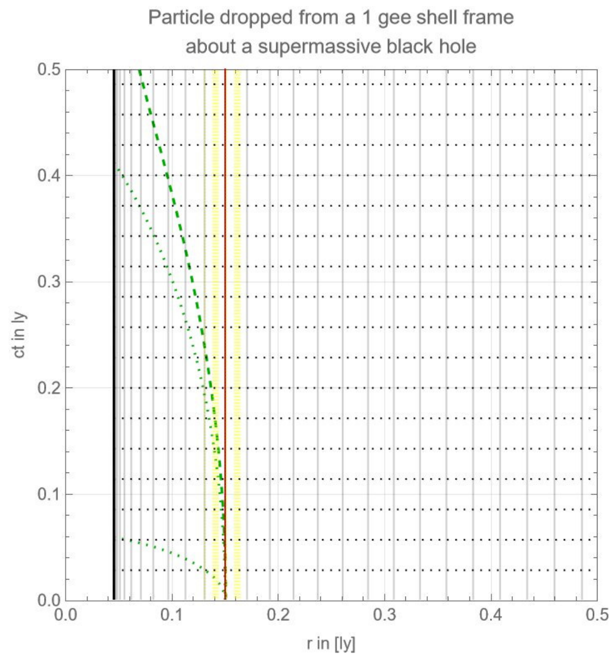


FIG. 7. Schwarzschild "far-time" r-ct view of a particle trajectory (green dashed) dropped from a 1 gee shell frame (red line) around a large supermassive (145 billion solar mass) non-rotating black hole with event horizon in dot-dashed blue.

in accelerating spaceships, and

$$W_{\rm esc} \simeq \left(\tfrac{\Delta t_{\rm far}}{\Delta\tau} - 1\right) mc^2 \overset{r\gg r_s}{\cong} G\frac{mM}{r} \tag{3}$$

for stationary surface dwellers above the surface of a massive sphere (like earth). This potential's gradient gives rise to the locally-approximate geometric forces discussed above. Non-local predictions require use of proper and coordinate time Lagrangians, as well as those previously-mentioned connection coefficients, but these are beyond the scope of this paper.

## VI. ENGINEERING NOMOGRAMS

Engineering nomograms which map the extended "proper coordinates" for an accelerated or curved-space time traveler allow one to visualize where traveler-point or flat-patch dynamics does and does not work, as well as to examine how extended definitions of simultaneity will apply to "arbitrary object trajectories" in terms of both bookkeeper and proper coordinates. The proper-coordinate overlay was e.g. discussed by Dolby and Gull[?] for an accelerated traveler, while arbitrary trajectories (e.g. of a dropped object) generally involve integration of coordinate-time and/or proper-time Lagrange equations. In (1+1)D quite accurate nomograms on flat screens and/or paper are thereby possible.

In order to illustrate how "flat-patch" or "traveler-point" dynamics works for accelerated frames and curved spacetimes, let's first consider views of an object dropped by an accelerated observer. Then we consider an object dropped by a shell-frame observer at fixed radius above a non-rotating gravitational mass.

Figure 6 offers x-ct diagrams for an observer undergoing constant proper acceleration $\alpha \simeq$ "one-gee" who drops an object at the "turnaround point" with respect to our flat spacetime bookkeeper frame, using two different definitions of extended simultaneity (i.e. models for the time and place of events non-local to our observer). The first (tangent free-float-frame simultaneity) only works in flat spacetimes as clocks can't otherwise be synchronized, while the latter (radar-time simultaneity[?] ) depends on future as well as past elements of our observer's trajectory.

Our accelerated observer's x,t coordinates relative to the map frame are described, in terms of proper time $\tau$ on the observer's clocks, by $t = (c/\alpha)\sinh[\alpha\tau/c]$ and $x = (c^2/\alpha)(\cosh[\alpha\tau/c]-1)$. Our dropped object's motion (which is at fixed $x_{\rm object}$ in the map frame) using the tangent free-float-frame definition then looks like:

$$\rho_{\rm object} \simeq -\frac{1}{2}\alpha\tau^2 - \frac{7\alpha^3}{24c^2}\tau^4 + O[\tau]^5, \tag{4}$$

while using radar time simultaneity it looks like:

$$\rho_{\rm object} \simeq -\frac{1}{2}\alpha\tau^2 - \frac{10\alpha^3}{24c^2}\tau^4 + O[\tau]^5. \tag{5}$$

The figures also show isocontours for $\tau$ and $\rho$ in case you want to examine other particle trajectories as well. Both highlight the "Rindler" event-horizon, which like the Schwarzchild event-horizon illustrated below prevents photons beneath it from ever catching our observer.

Regardless, the flat-patch or traveler-point description of our dropped object's motion is simply $\rho_{\rm object} \simeq -(1/2)\alpha\tau^2$, which is a good local approximation to both expressions above (and likely other definitions of extended simultaneity as well). The range of validity of this local approximation is denoted schematically by yellow shading along the worldline of our accelerated traveler.

Figure 7 shows a Schwarzschild "far-coordinate" r-ct diagram of trajectory models for a particle dropped from a "one-gee" shell frame outside a non-spinning supermassive (145 billion solar mass) black hole, whose mass has been chosen simply to make the event horizon radius a significant fraction of the shell radius as well. The figure shows isocontours for $\tau$ and $\rho$ only for radar-time simultaneity (since tangent-fff simultaneity doesn't work in curved spacetimes), again in case you want to examine other particle trajectories as well.

This plot here shows Schwarzschild geodesic trajectory of our dropped particle (dashed), which is asymptotic to (i.e. never crosses) the event horizon, as well as versions of the flat-patch or traveler-point trajectory $r_{\rm object} \simeq -(1/2)\alpha\tau^2$ (top dotted line) and the Newtonian approximation $\rho_{\rm object} \simeq -(1/2)\alpha\tau^2$ (bottom dotted line) where $\alpha = GM/r^2$.

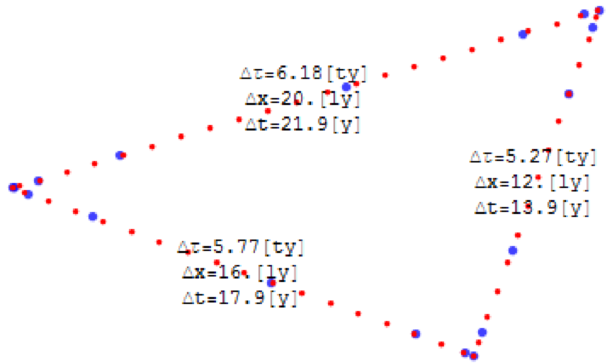


FIG. 8. Years elapsed in proper time (blue) and map time (red) on a 16+12+20 = 48 lightyear 1-gee acceleration round trip lasting about 17.2 traveler years and 53.7 map years.

As you can see, the flat-path or traveler-point trajectory is a much better approximation than the Newtonian one, even though its validity too is best in the shaded region close to our accelerated observer's world line.

## VII. DISCUSSION

The foregoing, summarized in Fig. 4, is targeted toward university physics teachers, but uses concepts which may be unfamiliar to some. In that context, we offer supplementary notes on introducing Newtonian time, kinematics, and dynamics as approximations to these more robust tools. Wentzel-Long et al are working on an even simpler "one-class" introducton to the Newtonian variables[?] , to at least correct the otherwise implicit assumption of time as global.

The traveler-point proper coordinate perspective on the importance of specifying "which clock", and on the distinction between proper and geometric forces with help from your cell-phone, can be helpful for students at the very beginning of their engineering physics education. It also provides "Newtonian-like" tools for the 3-vector addition of proper-velocities and momenta, as well as for proper-force/acceleration analysis at any speed using simultaneity defined by a free-float bookkeeper frame. Lot's of problems that are fun to think about might (cf. Fig. 9) emerge as a result.

The engineering nomograms discussed here, which quantitatively overlay book-keeper $x - ct$ and a single traveler's $\rho - c\tau$ curves, can be used to illustrate these observer's separate views of arbitrary events and trajectories in any model spacetime. In addition, (3+1)D simulations of accelerated motion between and around stars using simultaneity defined by bookkeeper far-time, but which are predicted on actions taken by a single traveler on their proper-time clock (cf. Fig. 8), can be used to model interstellar travel exactly if we ignore gravity, and approximately if one uses geometric-force gravity[?] or its proper-force approximation.

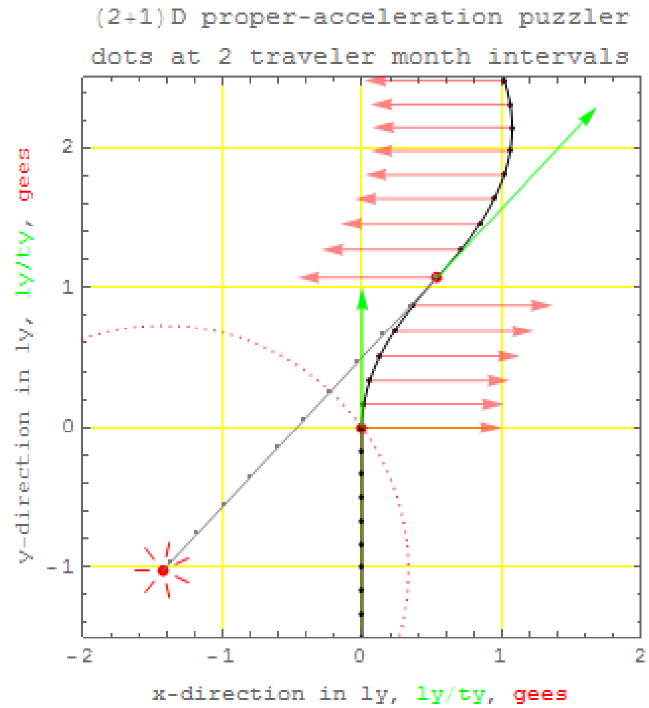


FIG. 9. A (2+1)D acceleration puzzler schematic of a starfleet battlecruiser traveling upward which, on seeing light from an enemy spaceship dropping out of hyperspace (dotted red circle), accelerates rightward in order to intercept. The enemy trajectory is in gray, while the starfleet cruiser trajectory is in black. Proper-time intervals are marked on both trajectories in 2 traveler-month intervals. Units for distance are lightyears, for proper-velocity vectors (green) are lightyears/traveler-year, and for proper-acceleration (red) is in lightyears/year. The battlecruiser then reverses acceleration direction in order to (eventually) recover its original trajectory, amazingly about four months of ship time **ahead** of schedule following a detour that took 4 years. A detour that saved time, at least for the traveler!

## VIII. CONCLUSION

In short, introductory physics texts might in the future introduce Newton's laws as approximations to "traveler-point" equations that work at any speed and (with help of local "geometric-force" approximations) in curved spacetimes and in accelerated frames. This would allow teachers to deconstruct Newton's implicit assumption of global time, and give students a simple path on their own to explore more extreme physics situations without having to spend time on those situations in the class itself.

It would also then make it natural to mention the roots of inertial forces and gravity (and "geometric forces" generally) in terms of differential aging, and of magnetism in terms of electrostatic charge combined with length contraction. This might whet potential interest in learning more, while keeping the focus on the Newtonian approaches at hand.

Moreover, the engineering nomograms discussed here can be used to illustrate how different observers see the same events and trajectories in spacetime. Finally, (3+1)D simulations of accelerated motion between and around stars can be used to model interstellar travel exactly *from the point of view of a single traveler* if one can ignore gravity.

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