Single-Frame Views of Flat or Local Space-Time

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(Dated: October 30, 2004)

Kinematic and dynamic relationships that follow from the flat-space metric equation are discussed in the context of a locally-referenced map-frame of co-moving yardsticks and synchronized clocks. These in effect allow one to adopt a local (or frame-specific) definition of simultaneity. In practice this may prove useful for relating observations to the experience of local-frame residents, even where more global strategies are needed to work out those predictions.

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I. 1D EQUATION SUMMARY

Minkowski's space-time version of Pythagoras' Theorem (the flat-space metric equation):

$$(cd\tau)^2 = (cdt)^2 - (dx)^2 \quad (1)$$

Useful Velocity Measures which arise because time's passage is frame-variant.

$$v \equiv \frac{dx}{dt},\ w \equiv \frac{dx}{d\tau},\ \gamma \equiv \frac{dt}{d\tau} \quad (2)$$

$$\eta \equiv \sinh^{-1}[\frac{w}{c}] = \tanh^{-1}[\frac{v}{c}] = \pm\cosh^{-1}[\gamma] \quad (3)$$

Velocity conversions which follow from the metric equation:

$$\gamma \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v}{c}^2}} = \sqrt{1+\frac{w}{c}^2} = \cosh[\eta] \quad (4)$$

Dynamically Conserved Quantities which depend on one's frame of reference:

$$p = mw;\ E = mc^2\gamma \quad (5)$$

Motion Integrals of Constant Proper Acceleration [1] (Felt by the Traveler):

$$\alpha = \frac{\Delta w}{\Delta t} = c\frac{\Delta\eta}{\Delta\tau} = c^2\frac{\Delta\gamma}{\Delta x} \overset{v\lll c}{=} \frac{\Delta v}{\Delta t} \overset{v\lll c}{=} \frac{1}{2}\frac{\Delta(v^2)}{\Delta x} \quad (6)$$

$$\Delta x = \frac{c^2}{\alpha}(\cosh[\frac{\alpha\Delta\tau}{c}+\eta_0]-\cosh[\eta_0]) \overset{v\lll c}{=} v_0\Delta t+\frac{1}{2}a(\Delta t)^2 \quad (7)$$

What is proper acceleration?

$$\alpha \equiv \sqrt{(\frac{dw}{d\tau})^2 - (c\frac{d\gamma}{d\tau})^2} = \frac{1}{\gamma}\frac{dw}{d\tau} \tag{8}$$

Relation between the (metric-invariant) Proper (Felt) and Coordinate (Frame-Variant) Acceleration:

$$\alpha \equiv \frac{1}{\gamma}\frac{dw}{d\tau} = \gamma^3 a, \text{ where } a \equiv \frac{dv}{dt} \tag{9}$$

At low speeds, the integrals reduce to:

$$\alpha = \frac{\Delta v}{\Delta t} = \frac{1}{2}\frac{\Delta(v^2)}{\Delta x} \tag{10}$$

II. 3D EQUATION SUMMARY

Minkowski's space-time version of Pythagoras' Theorem (the flat-space metric equation):

$$(cd\tau)^2 = (cdt)^2 - (d\mathbf{x} \bullet d\mathbf{x});\ \mathbf{u}\bullet\mathbf{u} \equiv u^2 = {u_\parallel}^2 + {u_\perp}^2 \tag{11}$$

Useful Velocity Measures which arise because time's passage is frame-variant.

$$v_\parallel \equiv \frac{dx_\parallel}{dt},\ v_\perp \equiv \frac{dx_\perp}{dt},\ w_\parallel \equiv \frac{dx_\parallel}{d\tau},\ w_\perp \equiv \frac{dx_\perp}{d\tau} \tag{12}$$

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt},\ \mathbf{w} \equiv \frac{d\mathbf{w}}{d\tau},\ \gamma \equiv \frac{dt}{d\tau},\ \gamma_\perp \equiv \frac{1}{\sqrt{1-(\frac{v_\perp}{c})^2}} \tag{13}$$

$$\eta_\parallel \equiv \sinh^{-1}[\frac{w_\parallel}{c}] = \tanh^{-1}[\gamma_\perp \frac{v_\parallel}{c}] = \pm\cosh^{-1}[\frac{\gamma}{\gamma_\perp}] \tag{14}$$

Velocity conversions which follow from the metric equation:

$$\gamma = \frac{1}{\sqrt{1-\frac{v}{c}^2}} = \sqrt{1+\frac{w}{c}^2} = \gamma_\perp \cosh[\eta_\parallel] \tag{15}$$

Dynamically Conserved Quantities which depend on one's frame of reference:

$$p = mw;\ E = mc^2\gamma \tag{16}$$

Motion Integrals of Constant Proper Acceleration (Felt by the Traveler, cf. [2]):

$$\alpha = \gamma_\perp \frac{\Delta w_\parallel}{\Delta t} = c\frac{\Delta \eta_\parallel}{\Delta \tau} = \frac{c^2}{\gamma_\perp}\frac{\Delta\gamma}{\Delta x_\parallel} \tag{17}$$

$$\alpha = \gamma_\perp \frac{\Delta w_\parallel}{\Delta t} = c\frac{\Delta \eta_\parallel}{\Delta \tau} = \frac{c^2}{\gamma_\perp}\frac{\Delta\gamma}{\Delta x_\parallel} \overset{v \lll c}{\equiv} \frac{\Delta v_\parallel}{\Delta t} \overset{v \lll c}{\equiv} \frac{1}{2}\frac{\Delta(v^2)}{\Delta x_\parallel} \tag{18}$$

What is proper acceleration?

$$\alpha \equiv \sqrt{(\frac{dw}{d\tau})^2 - (c\frac{d\gamma}{d\tau})^2} = \frac{\gamma_\perp}{\gamma}\frac{dw_\parallel}{d\tau} \tag{19}$$

Relation between the (metric-invariant) Proper (Felt) and Coordinate (Frame-Variant) Acceleration:

$$\alpha \equiv \frac{\gamma_\perp}{\gamma}\frac{dw_\parallel}{d\tau} = \frac{\gamma^3}{\gamma_\perp}a, \text{ where } a \equiv \frac{dv_\parallel}{dt} \tag{20}$$

At low speeds, the integrals reduce to:

$$\alpha = \frac{\Delta v_\parallel}{\Delta t} = \frac{1}{2}\frac{\Delta(v^2)}{\Delta x_\parallel} \tag{21}$$

How about a position-time integral...

$$\begin{bmatrix} c\Delta t \\ \Delta x_\parallel \\ \Delta x_\perp \end{bmatrix} = \frac{c^2}{\alpha}\begin{bmatrix} \gamma_\perp\left(\sinh\left[\frac{\alpha\Delta\tau}{c} + \eta_{\parallel o}\right] - \sinh\left[\eta_{\parallel o}\right]\right) \\ \cosh\left[\frac{\alpha\Delta\tau}{c} + \eta_{\parallel o}\right] - \cosh\left[\eta_{\parallel o}\right] \\ \gamma_\perp \frac{v_\perp}{c}\left(\sinh\left[\frac{\alpha\Delta\tau}{c} + \eta_{\parallel o}\right] - \sinh\left[\eta_{\parallel o}\right]\right) \end{bmatrix}, \tag{22}$$

where $\eta_{\parallel o}$ is simply the initial value for $\eta_\parallel \equiv \sinh^{-1}[\frac{w_\parallel}{c}]$.

At low speeds this becomes...

$$\begin{bmatrix} c\Delta t \\ \Delta x_\parallel \\ \Delta x_\perp \end{bmatrix} \overset{v \lll c}{\equiv} \begin{bmatrix} c\Delta\tau \\ v_{\parallel o}\Delta\tau + \frac{1}{2}\alpha\Delta\tau^2 \\ v_\perp\Delta\tau \end{bmatrix}, \tag{23}$$

III. TRACKING A WRT B AND C

A. Positions

When views of event A from two map-frames (B and C) are needed, the metric equation lets one relate x_{AC} and t_{AC} along respective time and space axes of C, to x_{AB} and t_{AB}, by equating $\{dx, cdt\}$ to: $\{\beta\gamma cdt', \gamma cdt'\}$ for time-like, and $\{\gamma dx', \beta\gamma dx'\}$ for space-like, increments in a moving primed frame. Here $\beta \equiv v/c$, subscript AB reads "of A with respect to frame B", and of course $v_{CB} = -v_{BC}$, $\gamma_{CB} = \gamma_{BC}$. You can let frames C and B share a common origin at time zero, or simply think of x, y and t variables as "change-in values" rather than absolute coordinates. Then if x is in the direction of v_{CB} and y perpendicular to it, the any-directional Lorentz transform follows:

$$x_{AC} = \gamma_{CB}(x_{AB} - t_{AB}v_{CB}) \tag{24}$$

$$y_{AC} = y_{AB} \tag{25}$$

$$t_{AC} = \gamma_{CB}(t_{AB} - x_{AB}v_{CB}/c^2) \tag{26}$$

B. Unidirectional velocities

Dividing the above equations by the frame-invariant τ_A, or equivalently multiplying by $\frac{\gamma_{AB}}{t_{AB}}$ then yields a set of proper-velocity and speed-of-map-time conversions. Here A is (typically) an object moving with respect to frames B and C, which in turn are in relative motion. In the proper-velocity equation, coordinate-velocities add while gamma factors multiply with subscript relationships which reduce to "vector addition" at low speeds (unit γ's). Useful e.g. to illustrate the many order-of-magnitude advantage of colliders over fixed-target accelerators.

$$w_{AC} = \gamma_{AB}\gamma_{BC}(v_{AB} + v_{BC}) \tag{27}$$

In the speed-of-map-time equation, gammas multiply but there is an extra velocity factor as well. Useful e.g. to derive Doppler red shifts when detector and source frames are separating rapidly.

$$\gamma_{AC} = \gamma_{AB}\gamma_{BC}(1 + \frac{v_{AB}}{c}\frac{v_{BC}}{c}) \tag{28}$$

Multiplying by mass converts the above equations into the "Einstein transform" equations for energy and momentum, while the addition rule for coordinate-velocities follows by dividing the previous equation into the one before it.

$$v_{AC} = \frac{(v_{AB} + v_{BC})}{1 + \frac{v_{AB}}{c}\frac{v_{BC}}{c}} \tag{29}$$

C. Anydirectional velocities

As above, A is often an object moving with respect to frames B and C in relative motion. If θ_{ABC} is the angle between v_{AB} and v_{CB} measured in the B frame, then we can define $v_{AB\|CB} = v_{AB}cos[\theta_{ABC}]$ and $V_{AB\perp CB} = v_{AB}sin[\theta_{ABC}]$. Then...

$$w_{AC\|CB} = \gamma_{AB}\gamma_{BC}(v_{AB\|CB} + v_{BC}) \tag{30}$$
$$w_{AC\perp CB} = \gamma_{AB}v_{AB\perp CB} \tag{31}$$

In the speed-of-map-time equation, gammas multiply but there is an extra velocity factor as well. These equations quickly yield things like the Doppler shift from a rapidly thrown flashlight, or the momentum and energy changes seen from rest when particles are ejected by a moving source.

$$\gamma_{AC} = \gamma_{AB}\gamma_{BC}(1 + \frac{v_{AB\|CB}}{c}\frac{v_{BC}}{c}) \tag{32}$$

Any-directional mass, energy, and coordinate-velocity equations follow, as outlined in the previous section.

IV. DYNAMICS

A. The frame-based Newtonian approach

As mentioned above, dynamically-conserved quantities that depend on one's frame of reference include:

$$\overleftarrow{p} = m\overleftarrow{w} \tag{33}$$

Here we've used a left-pointing arrow above the quantity to denote a column 3-vector. Below we introduce row 3-vectors (the transpose of the corresponding column 3-vector) with a right-pointing arrow, and 3x3 matrices with a double arrow, thus defining a useful matrix whose determinant is γ^2...

$$\overleftrightarrow{\Gamma} = \overleftrightarrow{1} + \frac{\overleftarrow{w}}{c} \bullet \frac{\overrightarrow{w}}{c} = \begin{pmatrix} 1 + \left(\frac{w_x}{c}\right)^2 & \frac{w_x}{c}\frac{w_y}{c} & \frac{w_z}{c}\frac{w_x}{c} \\ \frac{w_x}{c}\frac{w_y}{c} & 1 + \left(\frac{w_y}{c}\right)^2 & \frac{w_y}{c}\frac{w_z}{c} \\ \frac{w_z}{c}\frac{w_x}{c} & \frac{w_y}{c}\frac{w_z}{c} & 1 + \left(\frac{w_z}{c}\right)^2 \end{pmatrix} \tag{34}$$

Note that tail-to-tail arrows denote separate indices, while nose-to-nose arrows denote index sets which will cancel on multiplication. If we divide this matrix by it's determinant, we get:

$$\frac{\overleftrightarrow{\Gamma}}{\gamma^2} = \begin{pmatrix} 1 - \frac{v_y^2}{c^2} - \frac{v_z^2}{c^2} & \frac{v_x}{c}\frac{v_y}{c} & \frac{v_z}{c}\frac{v_x}{c} \\ \frac{v_x}{c}\frac{v_y}{c} & 1 - \frac{v_z^2}{c^2} - \frac{v_x^2}{c^2} & \frac{v_y}{c}\frac{v_z}{c} \\ \frac{v_z}{c}\frac{v_x}{c} & \frac{v_y}{c}\frac{v_z}{c} & 1 - \frac{v_x^2}{c^2} - \frac{v_y^2}{c^2} \end{pmatrix} \tag{35}$$

Following the usual practice of defining frame-variant force as the map-time rate of change of momentum, the relativistic version of Newton's 2nd Law then becomes (cf. [3])

$$\frac{\overleftarrow{F}}{m} = \gamma\overleftrightarrow{\Gamma}\overleftarrow{a} = \gamma_\perp \frac{\overleftrightarrow{\Gamma}}{\gamma^2}\overleftarrow{\alpha} = \alpha\gamma_\perp\left(\frac{1}{\gamma_\perp^2}\overleftarrow{i}_\| + \frac{v_\|}{c}\frac{v_\perp}{c}\overleftarrow{i}_\perp\right) = \frac{\alpha}{\gamma_t}\overleftarrow{i}_l \tag{36}$$

Here the symbol i represents a dimensionless unit vector. Subscripts for parallel ($\|$) and perpendicular ($\perp$) refer to component or unit vector directions with respect to the direction of accelerations α and a. Since force is not in the same direction as acceleration, we instead use the subscripts longitudinal (l) and transverse (t) to denote directions with respect to the net frame-variant force F. Finally, $\gamma_\perp \equiv 1/\sqrt{1-(v_\perp/c)^2}$, and $\gamma_t \equiv \sqrt{1+(w_t/c)^2}$. Using the inverse of the Γ matrix

$$\frac{1}{\overleftrightarrow{\Gamma}} = \overleftrightarrow{1} - \frac{\overleftarrow{v}}{c} \bullet \frac{\overrightarrow{v}}{c} = \begin{pmatrix} 1 - \left(\frac{v_x}{c}\right)^2 & \frac{v_x}{c}\frac{v_y}{c} & \frac{v_z}{c}\frac{v_x}{c} \\ \frac{v_x}{c}\frac{v_y}{c} & 1 - \left(\frac{v_y}{c}\right)^2 & \frac{v_y}{c}\frac{v_z}{c} \\ \frac{v_z}{c}\frac{v_x}{c} & \frac{v_y}{c}\frac{v_z}{c} & 1 - \left(\frac{v_z}{c}\right)^2 \end{pmatrix} \tag{37}$$

we can instead write in terms of unit vectors aligned with the force...

$$\overleftarrow{\alpha} = \frac{\gamma^3}{\gamma_\perp}\overleftarrow{a} = \frac{1}{\gamma_\perp}\frac{\gamma^2}{\overleftrightarrow{\Gamma}}\frac{\overleftarrow{F}}{m} = \frac{F}{m}\frac{1}{\gamma_\perp}\left(\gamma_t^2\overleftarrow{i}_l + \frac{w_\|}{c}\frac{w_\perp}{c}\overleftarrow{i}_t\right) = \frac{F}{m}\gamma_t\overleftarrow{i}_\| \tag{38}$$

Finally, converting between γ_t and $\gamma_\perp$ is facilitated by the identity...

$$\gamma_t\gamma_\perp = \frac{1}{\sqrt{\frac{1}{\gamma_\perp^4}+\left(\frac{v_\perp}{c}\frac{v_\parallel}{c}\right)^2}} = \sqrt{\gamma_t^4+\left(\frac{w_t}{c}\frac{w_l}{c}\right)^2} \tag{39}$$

In addition to the constant proper-acceleration integrals, we also have constant frame-variant force (or constant w_t) integrals...

$$\frac{F}{m} = \frac{\Delta w_\ell}{\Delta t} = c\frac{\Delta\eta_\ell}{\Delta\tau} = c^2\frac{\Delta\gamma}{\Delta x_\ell} \overset{v\ll c}{=} \frac{\Delta v_\parallel}{\Delta t} \overset{v\ll c}{=} \frac{1}{2}\frac{\Delta(v^2)}{\Delta x_\parallel} \tag{40}$$

where $\eta_\ell \equiv \ln[\frac{w_\ell}{c}+\sqrt{1+(\frac{w_\ell}{c})^2+(\frac{w_t}{c})^2}] = \sinh^{-1}[\frac{w_\ell}{c}+\frac{w_t^2}{2c(w_l+\gamma c)}]$ just as $\eta_\parallel \equiv \sinh^{-1}[\frac{w_\parallel}{c}] = \ln[\frac{w_\parallel}{c}+\sqrt{1+(\frac{w_\parallel}{c})^2}]$.

Newton's second law for flat spacetime looks like

$$\frac{\overleftarrow{F}}{m} = \gamma\left(\overleftrightarrow{1}+\frac{\overleftarrow{w}}{c}\bullet\frac{\vec{w}}{c}\right)\overleftarrow{a} = \frac{\alpha}{\gamma_t}\overleftarrow{i}_l \overset{v_\perp\ll c}{=} \overleftarrow{\alpha} \overset{v\ll c}{=} \overleftarrow{a} \tag{41}$$

where column three-vectors are denoted by left-pointing arrows, row three-vectors by right-pointing arrows, 3×3 matrices are denoted by double arrows, and unit vectors are denoted by the letter i. As usual v and a denote coordinate-velocity and coordinate-acceleration, while w and α denote proper-velocity and proper-acceleration, respectively. Also as usual $\gamma \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$ from the metric equation, while $\gamma_\perp \equiv \frac{1}{1-(\frac{v_\perp}{c})^2}$ and $\gamma_t \equiv \sqrt{1+(\frac{w_t}{c})^2}$. Vector components transverse and longitudinal to the frame-variant force direction are denoted by subscripts t and l, respectively, while components perpendicular and parallel to the frame-invariant proper-acceleration three-vector are denoted by subscripts $\perp$ and $\parallel$. This notation in mind, we can further convert between force and acceleration directions (where they differ signficantly) using

$$\overleftarrow{i}_l = \gamma_t\gamma_\perp\left(\frac{1}{\gamma_\perp^2}\overleftarrow{i}_\parallel+\frac{v_\perp}{c}\frac{v_\parallel}{c}\overleftarrow{i}_\perp\right) \tag{42}$$

and

$$\overleftarrow{i}_\parallel = \frac{1}{\gamma_t\gamma_\perp}\left(\gamma_t^2\overleftarrow{i}_l-\frac{w_t}{c}\frac{w_l}{c}\overleftarrow{i}_t\right). \tag{43}$$

$$\vec{F} = \frac{m\alpha}{\gamma_t}\vec{i}_l \overset{v_\perp\ll c}{=} m\vec{\alpha} \overset{v\ll c}{=} m\vec{a} \tag{44}$$

$$\sum_j \overleftarrow{F}_j = \frac{m\alpha}{\gamma_t}\overleftarrow{i}_l \overset{v_\perp\ll c}{\approx} m\overleftarrow{\alpha} \overset{v\ll c}{\approx} m\overleftarrow{a} \tag{45}$$

$$P \equiv \frac{dE}{dt} = \left(\sum\vec{F}\right)\bullet\overleftarrow{v} \tag{46}$$

B. Single-frame Biot-Savart

By way of application, imagine a wire running from left to right with a positive charge density of $+\frac{e}{\ell_o}$ and a negative charge density of $-\frac{e}{\ell_o}$, where ℓ_o is the distance between charges of given sign. On a chunk of wire of length ds, this translates to a positive charge $Q = \frac{e}{\ell_o}ds$ and a negative charge of $-\frac{e}{\ell_o}ds$ and therefore a net charge of 0. If the positive charges are stationary in the wire, but the negative charges are moving to the right with a speed v, we say that the current in the wire is $I = \frac{e}{\ell_o}v$, to the left. From Coulomb's law, the electrostatic force on a stationary test charge q a distance r above the wire is of course

$$F_{up} = F_+ + F_- = k\frac{qQ}{r^2} - k\frac{qQ}{r^2} = 0 \tag{47}$$

However if the test charge is moving to the right at a speed v, the flat-space version of Newton's 2nd Law predicts that the proper acceleration exerted on our test charge by the stationary positive charges *increases* by a factor of γ. Although we cannot analyze the frame-variant force between moving particles from the map frame being used here, by symmetry we can argue that the negative charges (which are now moving along with our test charge) must have had their contribution to the *frame-invariant* proper acceleration *decreased* by a factor of $\frac{1}{\gamma}$. The net proper acceleration experienced by our moving test charge is therefore non-zero. Likewise the net frame-variant force on our moving test particle, reduced from the net proper acceleration by that factor of γ, is also not zero. In other words...

$$F_{up} = \frac{1}{\gamma}(\gamma-\frac{1}{\gamma})k\frac{qQ}{r^2} = k\frac{qQ}{r^2}\frac{v^2}{c^2} = qv\left(\frac{k}{c^2}\frac{Ids}{r^2}\right) \tag{48}$$

One way to explain this force, due not to an electrostatic charge imbalance but to the frame dependence of forces in spacetime, is to say that the current creates a magnetic field B according to the Biot-Savart prescription in parentheses on the right, which in turn exerts a force according to the Lorentz Law $F = qvB$. Thus magnetic fields are a convenient tool for taking into account relativistic effects of the Coulomb force (most noticeable around neutral current-carrying wires), and the flat-space version of Newton's 2nd law facilitates the "single-frame" illustration of their relativistic roots.

V. FLAT 3D FIXED-COORDINATE METRICS

In the foregoing sections, we've tried to describe phenomena in terms of minimally variant quantities, and/or quantities defined with respect to a map-frame of co-moving and unaccelerated fiducial observers with synchronized clocks. Minkowski's metric equation, on the other hand, prepares us for describing motion in terms of any coordinate system, as well as in terms of curved coordinate systems for which extended map-frames do not exist.

In (3+1) dimensional spacetime, the general metric equation has the form

$$(cd\tau)^2 = -(ds)^2 = -g_{\alpha\beta}dx^\alpha dx^\beta. \quad (49)$$

where summation over indices (0,3) is implied when a common greek index appears both lowered and raised in the same product. The 4×4 tensor $g_{..}$ is called the metric tensor.

In this section, we look at how the metric equation changes as one simply transforms from one set of coordinates to another (e.g. spherical instead of cartesian coordinates) in describing one's location in a flat spacetime.

A. Minkowski's cartesian equation

This is Minkowski's classical space-time version of Pythagoras' theorem, applicable *locally* anywhere in our universe as far as we know:

$$(cd\tau)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2. \quad (50)$$

As indicated earlier, from this expressions follow for coordinate (dx/dt) and proper $(dx/d\tau)$ velocity, speed of map time $(dt/d\tau)$, coordinate and proper acceleration, the Lorentz transform between map frames, and (thanks to resulting local symmetries) conservation of relativistic momenergy as well. The corresponding metric tensor is simply

$$g_{..} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (51)$$

B. The polar coordinate metric equation

Here x and y are replaced by r and ϕ to give

$$(cd\tau)^2 = (cdt)^2 - (dr)^2 - (rd\phi)^2 - (dz)^2. \quad (52)$$

The metric tensor for $x^0 = ct$, $x^1 = r$, $x^2 = \phi$, and $x^3 = z$ is therefore

$$g_{..} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (53)$$

C. The spherical coordinate equation

Here x, y and z are replaced by r, θ and ϕ to give

$$(cd\tau)^2 = (cdt)^2 - (dr)^2 - (rd\theta)^2 - (r\sin[\theta]d\phi)^2. \quad (54)$$

The metric tensor for $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$ is therefore

$$g_{..} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin[\theta]^2 \end{pmatrix}. \quad (55)$$

D. Crystallography

The metric tensor is also helpful when dealing with translationally-periodic non-orthogonal coordinates. In this case, x, y and z are replaced by u, v and w, lattice vector components defined in terms of unit cell basis vectors a, b, and c, which make "lattice parameter" angles α between b and c, β between c and a, and γ between a and b. The direct metric tensor for coordinates ct, u, v and w is

$$g_{..} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & ab\cos[\gamma] & ac\cos[\beta] \\ 0 & ab\cos[\gamma] & b^2 & bc\cos[\alpha] \\ 0 & ac\cos[\beta] & bc\cos[\alpha] & c^2 \end{pmatrix}. \quad (56)$$

The unit cell volume is $\sqrt{\|g_{..}\|}$. Lattice vectors e.g. $v^\mu \equiv (0, u, v, w)$ can also be expressed in terms of reciprocal lattice co-vector components (i.e. Miller indices h, k and l) with help from $g_{\mu\nu}$ and its reciprocal $g^{\mu\nu}$ (cf. [4]). For example, the inter-planar spacing associated with reciprocal lattice covector $v_\alpha \equiv (0, h, k, l)$ is $d_{hkl} = \frac{1}{\sqrt{g^{\mu\nu}v_\mu v_\nu}}$, the corresponding lattice vector is $v^\mu = g^{\mu\nu}v_\nu$, and the interspot angle between two such covectors is $\theta_{12} = \cos^{-1}(\frac{g^{\mu\nu}v1_\mu v2_\nu}{|v1.||v2.|})$.

VI. ACCELERATED METRICS IN FLAT SPACE

The metric tensor also helps one consider descriptions of motion in moving and arbitrarily accelerated coordinate systems. The examples in this section involve accelerated frames in flat space-time. Coordinates referenced to a more familiar unaccelerated frame will be given the subscript o.

A. Einstein's rotating disk

This is a setup in which our fiducial (map-frame) observers find themselves rotating at angular velocity Ω along with a set of yardsticks arrayed around the circumference of a circle of radius r. In this case, the metric tensor for $x^0 = ct = ct_o\sqrt{1-(\frac{\Omega r}{c})^2}$, $x^1 = r = r_o$, $x^2 = \phi = \phi_o + \Omega t$, and $x^3 = z = z_o$ becomes

$$g_{..} = \begin{pmatrix} -\left[1-\left(\frac{\Omega r}{c}\right)^2\right] & 0 & \frac{\Omega r}{c}r & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\Omega r}{c}r & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (57)$$

As shown by Cook [5], the spatial metric which defines the local radar distance then assumes the decidedly non-Euclidean form

$$(d\ell)^2 = (dr)^2 + \frac{1}{1-\left(\frac{\Omega r}{c}\right)^2}(rd\phi)^2 + (dz)^2 \qquad (58)$$

Thus distance is the same in the radial direction as in a stationary frame, while circumferential yardsticks are length-contracted in the azimuthal direction. Hence the locally-measured perimeter increases according to azimuthal velocity Ωr, and contraction-effects within the frame of one azimuthal ring might even cause cracks!

Another interesting local insight is obtained by calculating the affine connection, and resulting geodesic equation, in terms of coordinates in this rotating frame. From this exercise, it follows that free objects experience a radial acceleration of the form

$$\frac{d^2r}{d\tau^2} = r\left(\gamma\Omega - \frac{d\phi}{d\tau}\right)^2, \qquad (59)$$

where γ is as usual $dt/d\tau$. Accelerations like this, which result from non-zero affine connection terms in the geodesic equation, are experienced as forces that act on every ounce of one's being. For observers at fixed ϕ, this "affine-connection force" is a relativistic version of the familiar centrifugal force felt as one goes around curves in a car. For observers with fixed r and changing ϕ e.g. for azimuthal track runners in a space station with artificial gravity, the above equation predicts a change in their "centrifugal weight" depending on how fast and in which direction they run. In particular by running quickly enough in a direction opposite to the satellite's rotation, they can make themselves weightless.

B. Rocket undergoing constant proper acceleration

Here we consider a collection of fiducial observers undergoing constant proper acceleration α in the x-direction. In terms of un-accelerated frame coodinates, we have $x^0 = ct = \frac{c^2}{\alpha}\sinh^{-1}[\frac{\alpha t_o}{c}]$, $x^1 = x = x_o + \frac{c^2}{\alpha}(1 - \sqrt{1+(\frac{\alpha t_o}{c})^2})$, $x^2 = y = y_o$, and $x^3 = z = z_o$. The metric tensor then becomes

$$g_{..} = \begin{pmatrix} -1 & \sinh[\frac{\alpha t}{c}] & 0 & 0 \\ \sinh[\frac{\alpha t}{c}] & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (60)$$

The non-Euclidean spatial metric that defines local radar distance, via contracted yardsticks, is

$$(d\ell)^2 = \left(\cosh[\frac{\alpha t}{c}]\right)^2 (dx)^2 + (dy)^2 + (dz)^2. \qquad (61)$$

Thus observer separations increase with local time away from the motion frame in which their clocks are synched.

C. Accelerating "flat space-time" convoy

Now consider the metric equation for coordinates in an extended map frame whose fiducial observers undergo constant proper accelerations in the x-direction, designed so as to keep the convoy's space-time flat [6]. In the standard form for this metric [7], all fiducial observers start from simultaneous rest in a Minkowski space-time, including a special reference observer of proper acceleration α who starts a distance $x_o = \frac{c^2}{\alpha}$ to the right of the coordinate system's origin and *event horizon*. Events in Minkowski space with $x_o < ct_o$ have no access to the convoy's space-time. The resulting convoy coordinates are $x^0 = ct = \frac{c^2}{\alpha}\tanh^{-1}[\frac{ct_o}{x_o}]$, $x^1 = x = \sqrt{x_o^2 - c^2t_o^2}$, $x^2 = y = y_o$, and $x^3 = z = z_o$, and the metric equation is simply

$$(cd\tau)^2 = (\frac{\alpha x}{c^2})^2(cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2. \qquad (62)$$

VII. STATIONARY CURVED SPACE METRICS

The unique value of metric tensors, of course, comes in describing curved space-time. Here we consider a couple of interesting statically curved space-time settings. The first allows us to describe gravitation around planets like ours, as well as around black holes, in terms of space-time curvature. The second allows us to imagine what a traversible wormhole might look like, if indeed such things are possible to create.

A. The Schwarzschild Black Hole

The metric tensor for $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$, here in "far coordinates", becomes

$$g_{..} = \begin{pmatrix} -(1-\frac{2GM}{c^2 r}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{c^2 r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin[\theta]^2 \end{pmatrix}. \qquad (63)$$

Strangely enough this space-time curvature, which only changes the metric coefficients by about a part per billion at the surface of the earth, beautifully explains much of what we know about gravity today.

In particular, this equation only applies exterior to a spherically symmetric mass M. For objects whose mass lies within the *event horizon* radius predicted by this metric, at $r = \frac{2GM}{c^2}$, the metric also only applies exterior to the event horizon as well. Cook [5] shows (strangely enough) that for both "shell frame" (r constant) and "rain frame" (free falling from infinity) observers, the local physical metric (i.e. Pythagoras' theorem) remains Euclidean, at least outside the event horizon. This in spite of the fact that the two are obviously in different states of acceleration. Is this also true for "orbit frames"?

Calculating far-coordinate affine connection terms, and the resulting geodesic equation, predicts a radial acceleration for stationary objects of the form

$$\frac{d^2r}{d\tau^2} = \frac{GM}{r^3}\left(r - \frac{2GM}{c^2}\right). \tag{64}$$

This reduces to Newton's gravity law for $r \gg \frac{2GM}{c^2}$.

B. Traversable Wormhole

Morris-Thorne [8] wormholes represent several chunks of spacetime "sewn together". In radial far coordinates, the external metric extends from wormhole mouth at radius a, out to infinity. For $r < a$, inward motion takes one into the wormhole throat until reaching $r = r_o$, at which point one begins to emerge from the throat into the universe on the other side. Here we describe an asymptotically flat wormhole whose exterior metric looks much like the Schwarzschild metric above.

For $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$ the interior ($r < a$) metric tensor is:

$$g_{\cdot\cdot} = \begin{pmatrix} -(1-(\frac{r_o}{a})^2) & 0 & 0 & 0 \\ 0 & \frac{1}{1-(\frac{r_o}{r})^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin[\theta]^2 \end{pmatrix}. \tag{65}$$

The exterior ($r > a$) metric is

$$g_{\cdot\cdot} = \begin{pmatrix} -(1-\frac{r_o^2}{ar}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{r_o^2}{ar}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin[\theta]^2 \end{pmatrix}. \tag{66}$$

This is one of several metrics considered in more detail by Lemos et al [9]. They find, for example, that a wormhole like that above, of 10 earth masses, would have throat radius $r_o = 10^6$ meters, a mouth radius a of 0.005 lightyears, and a traversal time of one year with peak (throat) velocity 0.01c.

VIII. MOVING CURVED SPACE METRICS

Lastly we turn to some simple curved space metrics which involve space-curving mass in motion. In reality, of course, all mass motions change the metric. Even in these examples, however, the effect of moving test particles is ignored.

A. Spinning Black Hole

Following Taylor and Wheeler [10], we look here for simplicity only at the equatorial plane of the extreme K-err metric, i.e. the metric for a black hole spinning at it's maximum possible rate. Using the same spherical "far coordinates" used in the Schwarzschild case, and defining "reduced-circumference" R by

$$R^2 \equiv r^2 + (\frac{GM}{c^2})^2 + \frac{2(\frac{GM}{c^2})^3}{r} \tag{67}$$

one gets for the metric tensor

$$g_{\cdot\cdot} = \begin{pmatrix} -(1-\frac{2GM}{c^2 r}) & 0 & \frac{4(GM/c^2)^2}{r} \\ 0 & \frac{1}{(1-\frac{2GM}{c^2 r})^2} & 0 \\ \frac{4(GM/c^2)^2}{r} & 0 & R^2 \end{pmatrix}. \tag{68}$$

B. An expanding universe

Here we consider de Sitter space-time, a special case of the set of Robertson-Walker expanding and contracting universes [1]. In spherical coordinates, the metric tensor looks like

$$g_{\cdot\cdot} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{2pt} & 0 & 0 \\ 0 & 0 & e^{2pt}r^2 & 0 \\ 0 & 0 & 0 & e^{2pt}r^2\sin[\theta]^2 \end{pmatrix} \tag{69}$$

where p is Hubble's constant. If $p = 0$, this reduces to the case of an unchanging flat space-time.

IX. DISCUSSION AND CONCLUSIONS

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