Teaching Newton with anticipation…

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Care making only clock-specific assertions about elapsed-time, and other ``space-time smart'' strategies from the perspective of a selected inertial map-frame, open doors to an understanding of anyspeed motion via application of the metric equation.

Keywords: space-time, acceleration, elapsed-time, lightspeed, kinematics; **Pac Numbers:** 03.30.+p, 01.40.Gm, 01.55.+b

Forward-Looking Introductions

Introductory texts tell students to specify a coordinate system whenever displacements are discussed. By asking that the clock's frame of motion be specified *whenever* time intervals are discussed, a habit is begun which will be needed to understand motion at high speeds.

For example, if we define a ``map'' frame in which both yardsticks and clocks reside, then velocity and acceleration defined using only map distances and times retain their classical form. In terms of these ``coordinate'' quantities, the classical equations of translational motion listed in Figure 1 make well-defined predictions in space-time. In fact, all except four of these predictions are exact not only for low-speed motion, but for motion at any speed as well!

Position $\vec{x}$ and time t are defined only with yardsticks *and clocks* fixed with respect to an arbitrary, but specified, inertial reference (or "map") frame.

Kinematic equations:

coordinate-velocity: $\vec{v} = \frac{d\vec{x}}{dt}$,

coordinate-acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$.

For constant *coordinate-acceleration* $\vec{a}$:

Galileo's velocity-time equation: $\vec{v} - \vec{v}_o = \vec{a}\Delta t$,

Galileo's velocity/distance equation: $\frac{1}{2}v^2 - \frac{1}{2}v_o^2 = \vec{a} \cdot \overrightarrow{\Delta x}$.

Dynamical equations:

Momentum $\boxed{\vec{p} \cong m\vec{v}}$ and energy $\boxed{E \cong \frac{1}{2}mv^2 + const.}$ are conserved in mechanical interactions. Their derivatives are force $\vec{F} \equiv \boxed{\frac{d\vec{p}}{dt} \cong m\vec{a}}$ and power $P \equiv \frac{dE}{dt} = \vec{F} \cdot \vec{v}$.

Velocity addition:

The velocity $\vec{v}_{32}$ of object 3 relative to frame 2, at low speed follows $\boxed{\vec{v}_{31} \cong \vec{v}_{32} + \vec{v}_{21}}$, if $\vec{v}_{31}$ and $\vec{v}_{21}$ are *wrt* frame 1.

Fig. 1: Classical equations written with variables defined unambiguously *at any speed*. Expressions accurate only at speeds much less than lightspeed are boxed. In these figures, the differential operator "d" is used like a "Δ" (which means final minus initial), applied in particular to tiny increments.

There is a caveat. Since these equations only concern distances and times measured from the vantage point of the map, they say *nothing* about the experiences of the accelerated object or traveler. More on this later. The utility of Fig. 1 for motion at any speed thus assumes that one avoids making the assumption, hidden or otherwise, that time passes similarly for everyones' clocks. Time elapsed is clock-dependent, save to first order at low speeds.

It is also worthwhile to look at *other* implicit assumptions of classical mechanics. Figure 2 lists many of the variables familiar from Newtonian physics, and asks the question: Which of these quantities, when used to describe motion between two sneezes of an airplane pilot (for example), did mechanics classically assume are independent of one's choice of reference (or map) frame?

Kinematic Parameters		
coordinate increments:	Map-distance traveled $\Delta\vec{x}$ in [m]	$\boxed{\text{Map-time elapsed } \Delta t \text{ in [s]}}$
first derivatives:		Coordinate-velocity $\vec{v} = \frac{d\vec{x}}{dt}$ in $[\frac{m}{s}]$
second derivatives:		$\boxed{\text{Coordinate-acceleration } \vec{a} = \frac{d\vec{v}}{dt} \text{ in } [\frac{m}{s^2}]}$
Dynamical Parameters		
motion integrals:	Momentum $\vec{p} \simeq m\vec{v}$ in $[kg\frac{m}{s}]$	Energy $E \simeq \frac{1}{2}mv^2 + const.$ in [j]
dynamical quantities:	Force $\boxed{\vec{F} = \frac{d\vec{p}}{dt} \simeq m\vec{a} \text{ in [N]}}$	Power $P = \frac{dE}{dt} = \vec{F} \bullet \vec{v}$ in [watt]

Figure 2: Quantities familiar in the classical study of motion. Outlined are quantities which are assumed (often implicitly) to be independent of one's choice of map frame. For motion at sufficiently high speeds, however, *none* of the quantities listed here is at all independent of one's frame of motion!

The answer is that distance between sneezes depends on map-frame (e.g. the pilot thinks both sneezes occur at the same place, namely the airplanes' cabin), as does velocity (since a map-frame moving with the pilot would see the pilot standing still). Momentum and energy obviously depend on velocity, and hence on choice of map-frame as well.

However, classical mechanics *often implicitly* assumes that elapsed times, observed accelerations, and applied forces are the same for all map-frames. Some of the teachers polled at two recent AAPT meetings also guessed that the rate at which energy increases is frame-independent classically. Rate of energy change of course depends on frame, since it equals force times velocity. So let's be explicit: For studies involving motion at speeds much less than the speed of light, of the quantities listed in Fig. 2 one can safely assume that only time-elapsed, observed acceleration, and applied force depend very little on one's choice of map-frame. For motion studies involving speeds approaching lightspeed, *all* of the quantities in Fig. 2 depend strongly on ones' choice of frame.

Forward-looking habits in the way introductory physics students think about time, as well as about how quantities will look in one frame or another, are important in at least two ways. First, they will minimize the things that students going on in physics will have to *unlearn*, like the use of time

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Minkowski's space-time version of Pythagoras' theorem relates elapsed traveler time $d\tau$ to map-coordinate changes via the metric equation: $(cd\tau)^2 = (cdt)^2 - (dx)^2$. Hence map-clocks are faster than traveler-clocks by the gamma-factor:

$$\gamma \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1-(v/c)^2}}.$$

Added kinematic equations that result:

proper-velocity: $w \equiv \frac{dx}{d\tau} = \gamma v$,

proper or "traveler–frame" acceleration: $\alpha = \frac{dw}{dt} = \gamma^3 a$.

For constant *proper-acceleration* α:

coordinate-time integral: $w - w_o = \alpha\Delta t$,

work-energy integral: $c^2(\gamma - \gamma_o) = \alpha\Delta x$,

rapidity integral: $c(\eta - \eta_o) = \alpha\Delta\tau$, where $\eta \equiv \sinh^{-1}[\frac{w}{c}]$.

Dynamical equations:

Momentum $p \equiv mw$; energy $E \equiv \gamma mc^2$;

variant force $F \equiv \frac{dp}{dt} = m\alpha = F_o$ (proper force)

and power $P \equiv \frac{dE}{dt} = Fv$.

For relative speeds:

Velocities add, γ's multiply: $w_{31} = \gamma_{31}v_{31} = \gamma_{32}\gamma_{21}(v_{32} + v_{21})$.

Fig. 3: Equations analogous to the classical equations, which *for unidirectional motion* are exact at any speed. The "non-defining equalities" complicate if motion is polydirectional. Traveler-time is introduced using the metric equation[1] as a space-time extension of Pythagoras' theorem while, in the spirit of J. Bell[2] and classic pedagogical tradition we stick to one reference frame with its tangible cartesian map and extended array of synchronized clocks. Frame-invariant proper-acceleration[3] is very simply expressable for unidirectional motion in context of a single map-frame[4]. **Note:** Comparing the rates of clocks (e.g. when defining γ as a "speed of map-time") requires that simultaneity, a property of the relationship between events that is strongly frame-dependent at high speeds, have specified meaning. Choice of a map-frame provides an unambiguous definition of simultaneity, namely that experienced by map-frame observers. However, the relative clock rates experienced by observers moving with respect to the map (like an accelerated traveler) will differ.

in a clock-independent manner. These things are crucial to a solid understanding of both special relativity and curved space-time. Second, as described below, such habits open the door to a deeper and simpler understanding of space-time for students with only one course in physics. This approach and philosophy is consistent with the proposal by Edwin Taylor in his 1998 AAPT Oersted Medal talk, which describes a way to provide deeper understanding with fewer math prerequisites in a second physics course. Empowering students in a *first course* with deeper physical intuition also provides us with a significantly better-informed taxpaying and consuming public.

Equations Good at Any Speed

Thus students can be taught to be wary of assumptions about frame invariance when talking about both times and distances, even as unidirectional motion is introduced via the usual classical expressions. Not presuming to understand how traveler clocks behave at high speeds, they may then be eager to learn more about velocity and acceleration at any speed, well before they are ready for ``multi-frame'' relativity. This may be done by introducing the metric equation as a space-time extension of Pythagoras' Theorem, written so that traveler-time is the invariant. This tells students most everything they need to know about traveler time at high speeds. However, more sophisticated use of this metric equation is needed before students are prepared to predict measurements with traveler yardsticks at high speed as well. Of course, saying ``don't go there'' could pique their interest in further studies of relativity and space-time geometry downstream.

Figure 3 introduces proper (or ``traveler'') time with the metric equation, followed by three corollary variables (gamma, proper velocity, and proper or ``felt'' acceleration). It then provides a set of equations, patterned after Fig. 1, which are *exact* for unidirectional motion at any speed. There is a new integral of constant proper acceleration (the so-called rapidity integral for proper time), and proper force F_o (defined as mass times proper acceleration) has been listed separately from frame-variant force F, since for non-unidirectional motion the two differ.

Lastly, Figure 4 now considers frame-invariance at any speed for the new variables as well as the variables discussed classically in Figure 2. Note that the new variables, which arose naturally from the relation for traveler-time provided by the metric equation, include three true frame-invariants: proper time, proper acceleration, and proper force. In a sense, then, the variables whose frame-independence was lost in the first part of this paper have been reborn in truly frame-invariant form, thanks to Minkowski's space-time extension of Pythagoras' theorem.

Kinematic Parameters			
coordinate increments:	Map-distance traveled $\Delta\vec{x}$ in [ly]	Map-time elapsed Δt in [y]	Traveler-time elapsed $\Delta\tau$ in [ty]
first derivatives:	Speed of map-time $\gamma = \frac{dt}{d\tau}$ in $[\frac{y}{ty}]$	Coordinate-velocity $\vec{v} = \frac{d\vec{x}}{dt}$ in $[\frac{ly}{y}]$	Proper-velocity $\vec{w} = \frac{d\vec{x}}{d\tau}$ in $[\frac{ly}{ty}]$
second derivatives:		Coordinate-acceleration $\vec{a} = \frac{d\vec{v}}{dt}$ in $[\frac{ly}{y^2}]$	Proper-acceleration $\vec{\alpha}$ in $[\frac{ly}{ty^2}]$
Dynamical Parameters			
motion integrals:	Momentum $\vec{p} = m\vec{w}$	Energy $E = \gamma mc^2$	
dynamical quantities:	Force $\vec{F} = \frac{d\vec{p}}{dt}$	Power $P = \frac{dE}{dt} = \vec{F} \bullet \vec{v}$	Proper-force $\vec{F}_o = m\vec{\alpha}$

Figure 4: Quantities useful in the study of motion at any speed, including three outlined variables whose value is *truly independent* of one's choice of map-frame! For problems at high speed on a human scale, convenient units for time are years [y] and traveler years [ty], while convenient units for distance are lightyears [ly]. In this case coordinate-velocity has units of *lightyears per map-year* or "c", while proper-velocity[5] (which has no upper limit[6]) is in *lightyears per traveler year*. One lightyear per traveler-year marks the transition to high speed nicely (it corresponds to v≅0.707c), but it is so cumbersome to say that a mnemonic (like "roddenberry") may help. For example, our "land-speed record" for m>0 objects is held by CERN electrons traveling at ≅10^5 [ly/ty]. Another advantage of such units of course is that, by happy coincidence, an acceleration of 1 "gee" (something we know humans can live with) is about 1 [ly/y^2].

Discussion

For students who are pictorially-oriented (or equation-shy), a nomogram plotting all variables versus distance traveled from rest can be put together with these equations. To illustrate, Figure 5 allows graphical solution of constant acceleration problems with almost any combination of input variables in range of the plot. As shown in the caption to Fig. 6, for *analytical* solution of problems the kinematic equations can be compacted into two simple equality strings. Other problems with solutions, on-line solvers, a discover-ityourself maze, and a companion derivation of the proper acceleration equation in Fig. 3, may be found on web pages linked to: http://www.umsl.edu/~fraundor/anyspeed.html.

Acknowledgments

This work has benefited indirectly from support by the U.S. Department of Energy, the Missouri Research Board, as well as Monsanto and MEMC Electronic Materials Companies. It has benefited most, however, from the interest and support of students at UM-St. Louis.

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[3] E. Taylor and J. A. Wheeler, *Spacetime Physics, 1st edition* (W. H. Freeman, San Francisco, 1963). The rapidity/proper-time integral was less discussed in the 1992 2nd edition of the book (E. F. Taylor, private communication), perhaps because teachers did not know how to incorporate it into more familiar multi-frame, relative-motion strategies for introducing kinematics at hi-speed.

[4] More derivations related to use of this strategy for describing map-based motion in (3+1)D are accessible through the Los Alamos e-print archives at http://xxx.lanl.gov/abs/physics/9704018.

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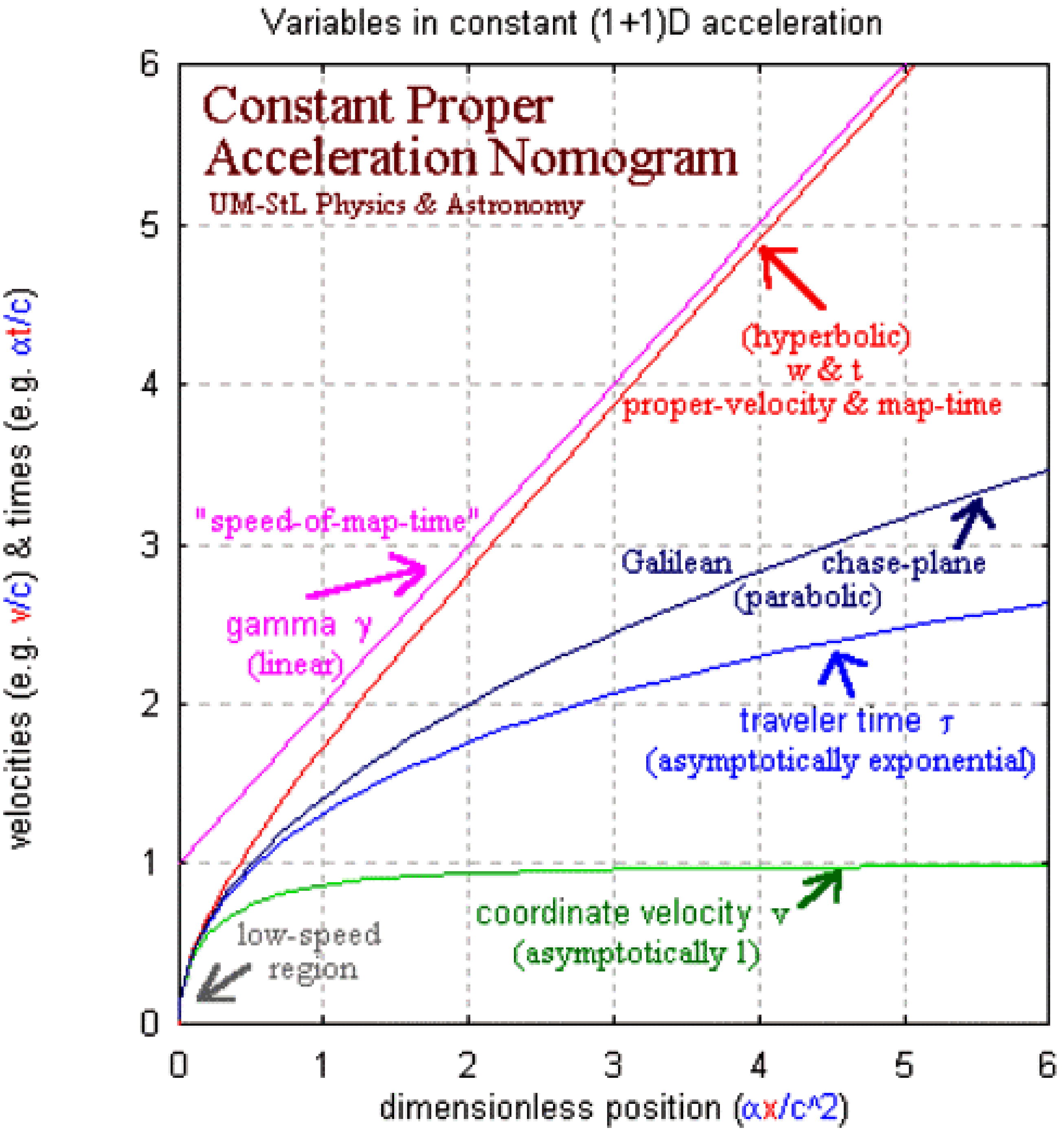

Figure 5: An "anyspeed" constant acceleration nomogram. If distance is in lightyears and time in years, then 1 "gee" proper-accelerations correspond to $\alpha = 9.8[m/s^2] \cong 1.03[ly/y^2] \cong 1[ly/y^2]$. Thus plotting a line up from the number 2 on the horizontal axis above allows one to determine, by inspection, the following results of a 1 "gee" trip from rest over a distance of 2[ly]: final coordinate velocity (≅0.9[ly/y]), elapsed traveler time (≅1.7[ty]), elapsed map time (≅2.8[y]), and final proper velocity (≅2.8[ly/ty]). The "chase-plane" parabola follows Galileo's low-speed curve (cf. Fig. 1).

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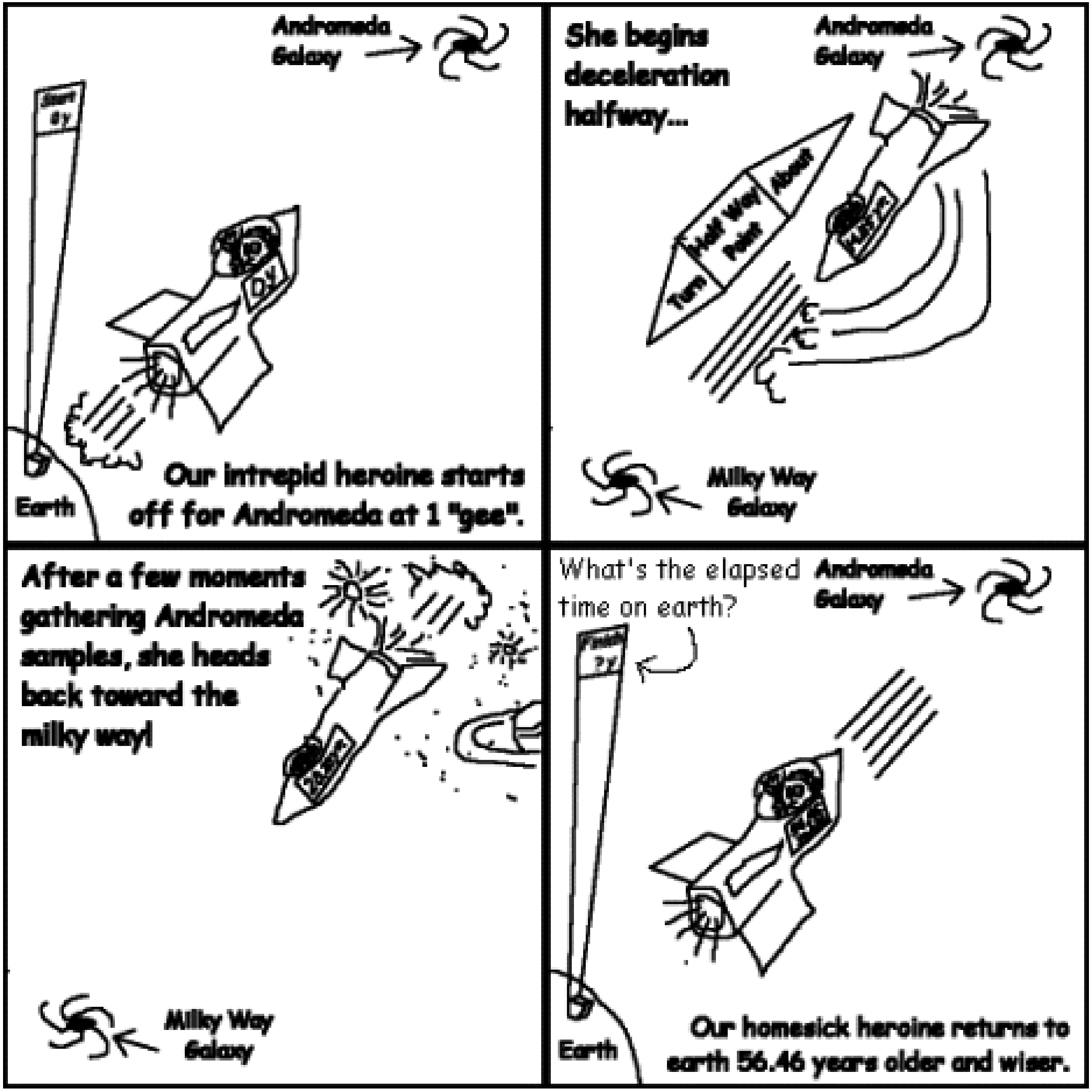

Figure. 6: Anyspeed kinematics for unidirectional motion are easily summarized by two equation strings. These are the <u>velocity conversions</u> **(1):** $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \sqrt{1+(w/c)^2} = \cosh[\eta]$ and the <u>motion integrals</u> **(2):** $\alpha = \frac{\Delta w}{\Delta t} = c\,\frac{\Delta\eta}{\Delta\tau} = c^2\,\frac{\Delta\gamma}{\Delta x}$. For the first of four legs of this trip, *proper-acceleration* α = 1 "gee" $\cong$ 1.03 [ly/y^2] and *traveler-time elapsed* $\Delta\tau = \frac{56.46}{4} = 14.115$ [ty]. Since speed starts at zero, from (2) *rapidity* $\eta_{final} = \Delta\eta = \frac{\alpha\Delta\tau}{c} \cong 14.538$. From (1), *proper-velocity* $w_{final} = c\,\sqrt{\cosh[\eta_{final}]^2-1} = c\,\sinh[\eta_{final}] \cong 1.03\times10^6$ [ly/ty], and from (2) *map-time elapsed* $\Delta t = \frac{w_{final}}{\alpha} = \frac{c}{\alpha}\sinh[\frac{\alpha\Delta\tau}{c}] \cong 10^6$ [y]. Combining all 4 legs gives total earth-time elapsed on return of 4×10^6 years, even though the traveler is only 56.46 years longer in the tooth!

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