A class period on spacetime-smart 3-vectors with familiar approximates

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The international Einstein-First initiative for modernizing the K-12 curriculum[1,2] may be the first comprehensive effort toward giving science students at all levels contact with 20th century insights, as followup e.g. to individual efforts with college physics textbooks like the "accelerator-first" text by Jonathon Reichert[3], and the "Six Ideas" series[4] by Thomas Moore. In the process, teachers will continue to find algorithmically simpler ways to introduce the concepts. In this paper we describe possible "one class period" use of a metric-first approach to describing motion and its causes, designed not to add new material for tests, but instead to highlight tools that both students *and teachers* might enjoy playing with <u>on the side</u> to improve their understanding of spacetime in the future.

Introductory physics students have Newton's laws drilled into their minds, but historically questions related to relativistic motion and accelerated frames have been avoided. With help from the metric equation, familiar 3-vector laws can be extended into the relativistic regime as long as one sticks with only one reference frame (to define position plus simultaneity), and considers something students are already quite familiar with, namely: motion using map-frame yardsticks as a function of time on clocks of the moving-object[5]. The question here is: How may one class period in an intro-physics class, e.g. as a preview before kinematics or later during a day on relativity-related material, be used to put the material we teach into a spacetime-smart context?

For example, consider driving a car: When you look at your speedometer, what are you seeing? The reported speed is not relative to some inertial frame with synchronized clocks on the side of the road, since speedometers use the rotation of the wheels[6] (which make static contact with the road) per unit time on the car's on-board clocks. This "proper" ratio of map distance $\Delta\mathbf{x}$ to traveler time $\Delta\tau$ at any speed (e.g. even if lightspeed as for Mr. Tompkins[7] was only ≈2.5 mph) turns out to be proportional to 3-vector momentum $\mathbf{p}$, to have no upper limit, and to also be most simply related to kinetic energy and driver/pedestrian reaction times. It reduces to map distance $\Delta\mathbf{x}$ per unit map time Δt only at low speeds.

Another remarkable everyday example of the "traveler point" approach is the fact that your phone accelerometer cannot detect gravity, as shown in Fig. 1. It only detects the normal force which prevents us from falling through the floor. It also fails to detect inertial forces, like those which push you back into the seat (or to the outside of a curve) when your car accelerates (or follows a curved path). This is good news, coming from general relativity, which says that our accelerometer only detects proper forces but that the "undetected" class of geometric forces (associated with accelerated frames or curved spacetime) can in general be approximated locally as if they are one (or more) proper forces. This honored tradition, of treating geometric forces as proper, was of course started in the 17th century by none other than Issac Newton himself.

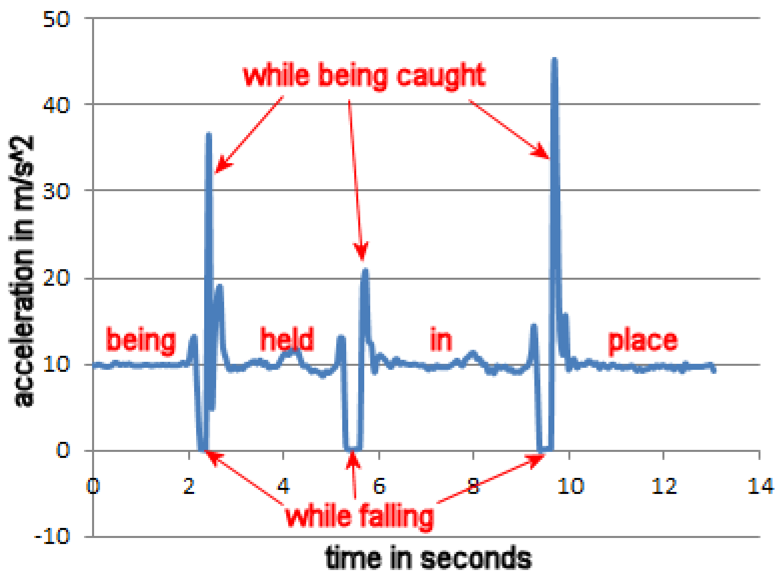

Figure 1: Accelerometer data from a phone dropped and caught 3 times, captured wsith Google Science Journal.

For example, in our college algebra-based "basic physics II" class, Walker's text[8] has a chapter near the end on relativity. Our strategy is to introduce the book's, along with traveler-point, tools for dealing with problems of time dilation, unidirectional velocity addition, and relativistic energies/momenta. Length contraction is off the table, because it requires two extended frames with synchronized clocks. Proper-velocity **w** = γ**v** and restmass m is used instead of "relativistic mass" to preserve the standard relationship between momentum and velocity, and students are only being asked to master that subset of problems posed in the book which can solved with or without these "hybrid kinematic" tools, as shown in the table below.

Table I: Newton at any speed, using equations from $(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2$

Quantity\Variable	**standard offering**	**traveler-point version**	**low-speed version**
time dilation $\gamma \equiv \Delta t/\Delta\tau$	$\gamma = 1/\mathrm{Sqrt}[1-(v/c)^2]$	$\gamma = \mathrm{Sqrt}[1+(w/c)^2]$	$\gamma \approx 1$
relative velocities $v \equiv \Delta x/\Delta t$; $w \equiv \Delta x/\Delta\tau$	$v_{ac} = (v_{ab}+v_{bc})/(1+v_{ab}v_{bc}/c^2)$	$w_{ac} = \gamma_{ab}\gamma_{bc}(v_{ab} + v_{bc})$	$v_{ac} \approx v_{ab} + v_{bc}$
momentum	$p = m\gamma v$	$p = mw$	$p \approx mv$
total energy	$E = \gamma mc^2$	$E = \gamma mc^2$	$E \approx mc^2$
kinetic energy	$K = (\gamma-1)mc^2$	$K = (\gamma-1)mc^2$	$K \approx \frac{1}{2}mv^2$

More generally we suggest initial mention (even if only in passing) of the "traveler-point variables" (chosen because they either have frame-invariant magnitudes or because they don't require synchronized clocks), namely traveler or *proper time* τ, *proper velocity* defined as map distance per unit traveler time $\mathbf{w} \equiv \Delta\mathbf{x}/\Delta\tau$, and the net *proper force* $\Sigma\boldsymbol{\xi} = m\boldsymbol{\alpha}$ felt by on-board accelerometers. These are approximated at low speeds by the more familiar map time t, coordinate velocity $\mathbf{v} \equiv \Delta\mathbf{x}/\Delta t$, and net map-based force $\Sigma\mathbf{f} = \Delta\mathbf{p}/\Delta t$. By sticking with displacements $\Delta\mathbf{x}$ and simultaneity defined by a single bookkeeper or map reference frame (i.e. the metric), as shown in Table II we can simply describe time-dilation $\gamma \equiv \Delta t/\Delta\tau$ and constant unidirectional proper acceleration $\boldsymbol{\alpha}$ at any speed, even when there's no time to explore 3-vector proper velocity/acceleration or multi-frame phenomena like length contraction.

The unidirectional proper-velocity addition equation given in Table II, for example, allows students to see the “collider advantage” in more visceral terms, which may even fire up the imagination of NASCAR fans as depicted in the relative velocity illustration of Figure 2 (inspired by an XKCD cartoon). Similarly the unidirectional equations of constant proper acceleration given in Table II allow students to easily calculate the map and traveler times elapsed on constant proper-acceleration round trips between stars, as illustrated in Figure 3.

Table II: Traveler-point dynamics in (1+1)D flat spacetime.
Conserved energy $E = \gamma mc^2$, momentum $\mathbf{p} = m\mathbf{w}$, aging-factor $\gamma \equiv \delta t/\delta\tau$, proper-velocity $\mathbf{w} \equiv \delta\mathbf{r}/\delta\tau \equiv \gamma\mathbf{v}$, coordinate acceleration $\mathbf{a} \equiv \delta\mathbf{v}/\delta t$;
In the first 4 rows, τ is traveler-time from "rest" with respect to the map frame, and $\boldsymbol{\alpha}$ is a fixed space-like proper-acceleration vector.
*Asterisk means that the (1+1)D relation also works in (3+1)D.

relation	**w<<c**	**(1+1)D**
map time elapsed t	$t \approx \tau$	$t = c/\alpha \sinh[\alpha\tau/c]$
map displacement **r**	$\mathbf{r} \approx \mathbf{v}_o t + \frac{1}{2}\mathbf{a}t^2$	$x = (c^2/\alpha)(\cosh[\alpha\tau/c]-1)$
aging factor $\gamma \equiv \delta t/\delta\tau$	$\gamma \approx 1 + \frac{1}{2}(v/c)^2$	$\gamma = \cosh[\alpha\tau/c]$
proper velocity $\mathbf{w} \equiv \delta\mathbf{r}/\delta\tau$	$\mathbf{w} \approx \mathbf{v} \approx \mathbf{v}_o + \mathbf{a}t$	$w = c \sinh[\alpha\tau/c]$
relative velocity $\mathbf{w}_{AC}$	$\mathbf{v}_{AC} \approx \mathbf{v}_{AB} + \mathbf{v}_{BC}$	$w_{AC} = \gamma_{AB}\gamma_{BC}(v_{AB}+v_{BC})$
*momentum **p**	$\mathbf{p} \approx m\mathbf{v}$	$\mathbf{p} = m\mathbf{w} = m(\gamma\mathbf{v})$
*energy E	$E \approx mc^2+\frac{1}{2}mv^2$	$E = mc^2+K = \gamma mc^2$
felt (ξ) ↔ map-based (**f**) force conversions	$\mathbf{f} \approx \boldsymbol{\xi}$	$f = \xi$
*work-energy	$\delta E \approx \Sigma\mathbf{f}\bullet\delta\mathbf{r}$	$\delta E = \Sigma\boldsymbol{\xi}\bullet\delta\mathbf{r}$
*action-reaction	$\mathbf{f}_{AB} = -\mathbf{f}_{BA}$	$\mathbf{f}_{AB} = -\mathbf{f}_{BA}$
*map-based force (**f**)-momentum (**p**) *felt force (ξ)-acceleration ($\boldsymbol{\alpha}$)	$\Sigma\mathbf{f} = \delta\mathbf{p}/\delta t \approx m\mathbf{a}$	$\Sigma\mathbf{f} = \delta\mathbf{p}/\delta t$ $\Sigma\boldsymbol{\xi} = m\boldsymbol{\alpha}$

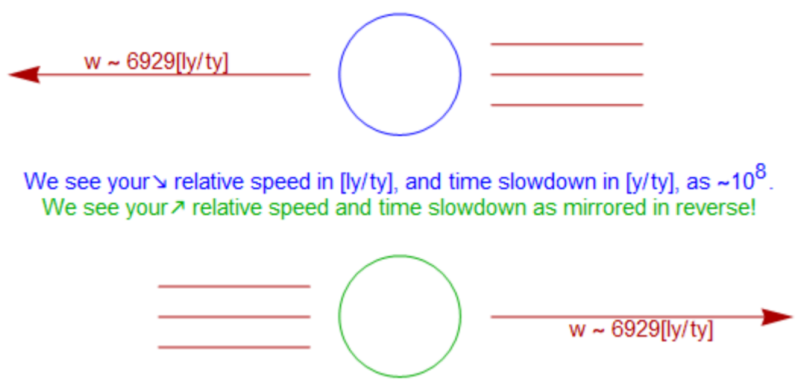

Figure 2: Two 6.5 TeV LHC protons send messages to each other, while passing at proper velocities of about ≈ 6929 lightyears/traveler-year, for a collider energy advantage of Krel/K ≈ 13,859 times the energy of stationary target collision.

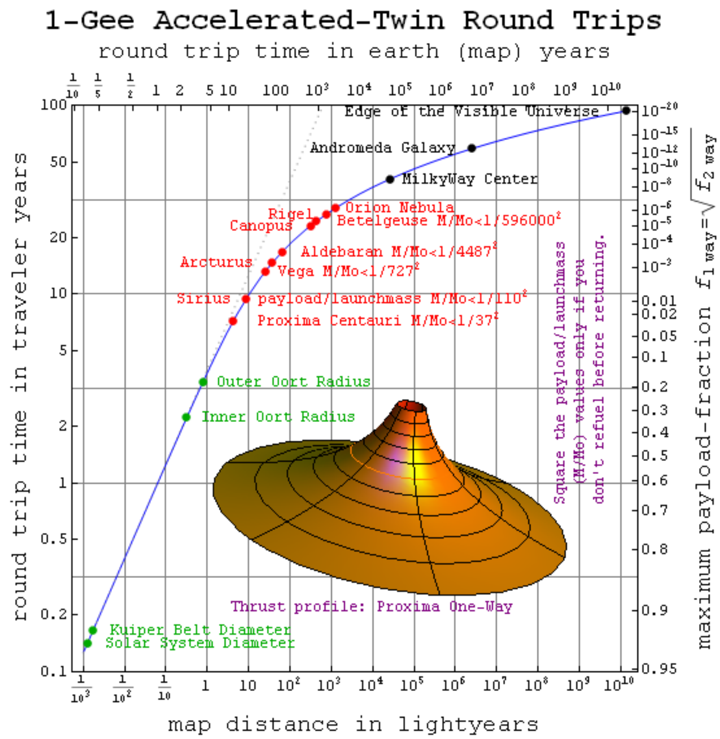

Figure 3: Round trip times and a sample thrust profile for a spaceship capable of constant 1-gee acceleration and avoiding collision with atoms.

In passing, we should also mention the curious relationship between various energies and the time-dilation or differential-aging factor $\gamma = \Delta t/\Delta\tau$. The basic relationship, given in Table II, allows us to say that kinetic energy of motion *with respect to inertial frames* in flat spacetime is $K = (\gamma-1)mc^2$. Remarkably, however, in curved space time and in accelerated frames, relations like this also express potential energies associated with geometric forces. This is easiest to see when standing in an artificial (centrifugal) gravity well, where the kinetic energy from a fixed external point of view looks like a potential energy well of depth $U \approx \frac{1}{2}m\omega^2r^2$ to the rotating inhabitant. However, it also turns out to be true in a spaceship of length L undergoing constant proper acceleration α, where $\Delta U = (\Delta t_{leading}/\Delta t_{trailing}-1)mc^2 \approx m\alpha L$, and in the gravity of a non-spinning sphere of mass M and radius R, where the escape energy for mass m on the surface (when R is much more than the Schwarzschild radius) is $W_{esc} = (\Delta t_{far}/\Delta\tau-1)mc^2 \approx GMm/R$. This and the kinetic differential-aging factors must, for example, both be considered when calculating your global-positioning-system location.

Cautions for "traveler-point dynamicists", especially when considering the vantage point of more than one "map-frame" or bookkeeper metric:

- 1^{st} caution: Specify "which clock" when talking about time elapsed, and which "map frame" when talking about position.
- 2^{nd} caution: Try to stick with a single map frame of yardsticks and bookkeeper or "metric time" clocks. This takes discussion of length contraction and Lorentz transforms (both requiring two extended frames) off the table, but allows 3-vector dynamics to be added.

- 3rd caution: Like rates of energy change at any speed, map-based forces (magnitude & direction) differ from one frame to the next at high speeds, even if the frames are only moving at a constant speed with respect to one another. This frame dependence actually gives rise to a kinetic versus static breakdown of all proper forces, whose usefulness in the case when there are "oppositely charged" force-carriers is behind the 19th century distinction between magnetic and electrostatic fields.
- 4th caution: The simultaneity of separately located events is also frame dependent, i.e. differently moving observers may disagree on which of two "space-like separated" events came first, just as the filial ordering of non-descendant relatives in a family tree may disagree on an individual's generation[9].
- 5th caution: Relative 3-vector proper-velocity addition (e.g. between co-moving reference frames) is possible, but may be complicated by both "clock changes" which affect component magnitudes, and by changes in the reference metric (which affect both component magnitudes and directions).
- 6th caution: If energy is not conserved in an interaction between objects traveling at high speeds, momentum may not be either since differently-moving frames allow trades between energy E and momenta **p** (as well as between motion-through-time $\delta t/\delta\tau$ and motion-through-space $\delta\mathbf{x}/\delta\tau$) because only a sum of both, i.e. $c^2 = c^2(E/mc^2)^2 - (\mathbf{p}/m)^2 = c^2(\delta t/\delta\tau)^2 - (\delta\mathbf{x}/\delta\tau)^2$), is frame invariant.
- 7th caution: Geometric forces like gravity and centrifugal in general only work locally, i.e. in regions within which your reference spacetime metric is "locally flat". Extensions are possible, e.g. with tidal and Coriolis forces, by combining forces from separate regions.

To provide space for discussing sample problems, and for the development of on-line calculators and simulators to further empower students and introductory teachers with this metric-first[10] or "one-map two-clock"[11] approach, we've created some space up here on google sites for further discussion.

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