A traveler-centered intro to kinematics

P. Fraundorf
*Physics & Astronomy/Center for NanoScience, U. Missouri-StL (63121)**

(Dated: March 26, 2014)

Treating time as a local variable permits robust approaches to kinematics that forego questions of extended-simultaneity, which because of their abstract nature might not be addressed explicitly until a first relativity course and even then without considering the dependence of clock-rates on position in a gravitational field. For example we here use synchrony-free "traveler kinematic" relations to construct a brief story for beginning students about: (a) time as a local quantity like position that depends on "which clock", (b) coordinate-acceleration as an approximation to the acceleration felt by a moving traveler, and (c) the geometric origin (hence mass-independence) of gravitational acceleration. The goal is to explicitly rule out global-time for all from the start, so that it can be returned as a local approximation, while tantalizing students interested in the subject with more widely-applicable equations in range of their math background.

PACS numbers: 05.70.Ce, 02.50.Wp, 75.10.Hk, 01.55.+b

CONTENTS

I. INTRODUCTION

An emergent cross-disciplinary objective of Bayesian model-selection is the search for models that are *least surprised* by incoming data, because this goal at once focuses on both algorithmic simplicity and goodness of fit[1,2]. The task of content-modernization in physics education research might, in theory, also be inspired by this objective. In that context this is a paper on kinematics content-modernization. The immediate goal is only to suggest a minor shift in our introduction to the subject, with hopes that future *experimental* physics education research can explore its impact on the physics experience of future physics-majors, as well as on the much larger population of future non-majors.

Because introductory physics sequences are often taught by multiple instructors at the same school, it is not easy to adopt texts which involve radically different approaches. Hence progress in modernizing the approach has been slow, even though the introductory series comprises the last course contact that physics departments have with the lion's share of the future consumers and voters that they encounter. Since modern physics is increasingly relevant to everyday life, this is an increasingly important opportunity missed.

The problem has been addressed variously in recent decades for example by Jonathan Reichert's seven-chapter particle-physics introduction to unidirectional motion[3], by Tom Moore's Six Ideas text[4] which has e.g.

at Washington University in St. Louis spawned a parallel introductory-physics sequence, and by Louis Bloomfield's How Things Work text[5] designed to look like a collection of everyday topics to the students while faculty can choose to see it as a standard-format two-semester sequence, with modern physics insights, squeezed into a one-semester course with few math requirements. Although these are important initiatives, the standard-format approach retains a significant market-share.

Given the standard-format approach and limited time, individual teachers and text authors may nonetheless work to minimize the cognitive-dissonance between traditional approaches and modern insights by integrating those historical approaches into a modern context from the start. In statistical physics, for example, this might involve starting from the statistical roots of concepts like temperature and entropy[6,7] i.e. the approximations underlying the ideal gas law and equipartition, as has been done in senior undergrad texts for many decades[8–11]. In quantum applications it might involve mention of "explore all paths" logic[12] and subsystem-correlations before moving to energy-eigenvalue stationary states. In modern physics generally it might involve illustrations of how the value and application of a given concept (like position in the subatomic world, or time on cosmic scales) can change significantly from one size-scale to the next[13].

It is the much more modest objective of this paper to suggest a few paragraphs to offer at the beginning of the section on unidirectional kinematics. These paragraphs will introduce Galilean kinematics and Newtonian gravity as the approximations that they are, while placing a weak spot in everyone's implicit assumption of global time. Thanks to synergy between relativistic insights at all levels, this opens the door to a more integrative perspective on everyday topics like the perspective from accelerated frames, while offering to ambitious students a range of other types of problems with which they can play from the vantage point of a single bookkeeper or map frame[14–16].

This paper therefore provides a "kinematics teaser" as one possible format for those opening paragraphs, followed by a survey of complementary problem-types that may pique student interest but which will be largely ignored. That's because with or without a more modernized context, classes from most of today's standardized texts have much business-as-usual to cover.

In particular here we argue that intro-physics students in an engineering-physics class might augment their introduction to unidirectional-kinematics with a technical story that's in harmony with the general trend toward metric-first approaches[17–19]. This might help nip the implicit Newtonian-assumption of universal time in the bud. It would also introduce: (i) momentum-proportional proper-velocities[20,21] that can be added vectorially at any speed[22], and (ii) proper-acceleration[23] in harmony with the modern (equivalence-principle based) understanding of geometric-accelerations i.e. accelerations that arise from choice of a non free-float-frame coordinate-system.

In the process of putting into context the Newtonian-concepts of: **reference-frame**, **elapsed-time** t, **position** x, **velocity** v, **acceleration** a, **constant-acceleration-integral** and **gravitational-acceleration** g, this intro is designed to give students a taste of the more robust technical-concepts highlighted in **bold** below. If these engender critical-discussion (rather than a focus on intution-conflicts), all the better as such concepts might help inspire the empirical-scientist inside students even if they never take another physics course.

The few take-home equations from this introduction that students will likely be asked to master in an intro course will be highlighted in red. Hence you might simply consider these notes an alternate, but fun, intro to the Galilean kinematic relationships for tracking motion and its causes (including the assumption of global time) that are often just presented or assumed *fait accompli*.

II. TIME AS MERELY LOCAL

In the first part of the 20th century it was discovered that time is a local variable, linked to each clock's location through a space-time version of Pythagoras' theorem i.e. the local metric equation. Both height in the earth's gravitational field, and clock-motion, affect the rate at which time passes on a given clock. Both of these effects must, for example, be taken into account in the algorithms used by handheld global positioning systems.

The fact that time is local to the clock that's measuring it means that we should probably address the question of extended-simultaneity (i.e. when an event happened from your perspective if you weren't present at the event) only as needed, and with suitable caution. Care is especially needed for events separated by "space-like" intervals i.e. for which $\Delta x > c\Delta t$ where c is the space/time constant sometimes called "lightspeed".

Recognizing that traveling clocks behave differently also gives us a synchrony-free[21] measure of speed with minimal frame-dependence, namely proper-velocity[20] $\vec{w} \equiv d\vec{x}/d\tau = \gamma\vec{v}$, which lets us think of momentum as a 3-vector proportional to velocity regardless of speed. Here τ is the frame-invariant proper-time elapsed on the traveling object's clock, Lorentz-factor $\gamma \equiv dt/d\tau$, and as usual coordinate-velocity $\vec{v} = d\vec{x}/dt$.

Recognition of the height-dependence of time as a kinematic (i.e. metric-equation) effect, moreover, allows us to explain the fact that free-falling objects are accelerated by gravity at the same rate regardless of mass. Hence gravity is now seen as a geometric force instead of a proper one, which is only felt from the vantage point of non "free-float-frame" coordinate-systems like the shell-frame normally inhabited by dwellers on planet earth.

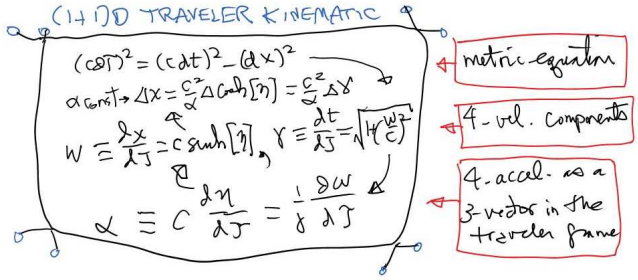


FIG. 1: The synchrony-free traveler-kinematic in (1+1)D.

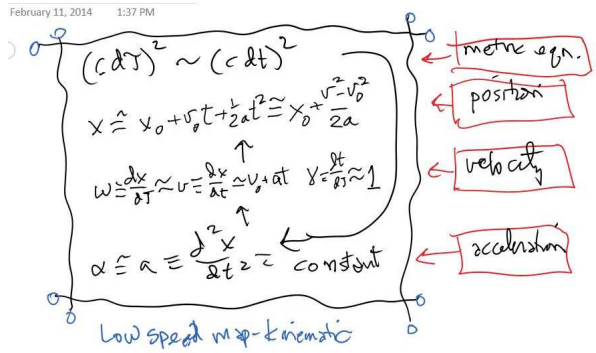


FIG. 2: Galileo's approximation to the traveler-kinematic.

III. KINEMATICS TEASER

The world is full of motion, but describing it (that's what kinematics does) requires two perspectives: (i) the perspective of that which is moving e.g. "the traveler", and (ii) the reference or bookkeeper perspective which ain't moving e.g. "the map". Thus at bare minimum we imagine a map-frame, defined by a coordinate-system of yardsticks say measuring map-position x with synchronized-clocks fixed to those yardsticks measuring map-time t, plus a traveler carrying her own clock that measures traveler (or proper) time τ. A definition of extended-simultaneity (i.e. not local to the traveler and her environs), where needed for problems addressed by this approach, is provided by that synchronized array of map-clocks.

We will assume that map-clocks on earth can be synchronized (ignoring the fact that time's rate of passage increases with altitude), but let's initially treat traveler-time τ as a local quantity that may or may not agree with map-time t. The space-time version of Pythagoras' theorem says that in flat space-time, with lightspeed constant c, the **Lorentz-factor** or "speed of map-time" is $\gamma \equiv dt/d\tau = \sqrt{1+(dx/d\tau)^2/c^2}$. This indicates that for many engineering problems on earth (except e.g. for GPS and relativistic-accelerator engineering) we can ignore clock differences, provided we *imagine* further that gravity arises *not* from variations in time's passage as a function of height (i.e. from kinematics) but from a dynamical force that acts on every ounce of an object's being. In that case we can treat time as global, and imagine that accelerations all look the same to observers who are not themselves being accelerated.

Before we take this leap, however, we might spend a paragraph describing kinematics in terms of traveler-centered variables (Fig. 1) that allow one to describe motion locally regardless of speed and/or space-time curvature. These variables are frame-invariant **proper-time** τ on traveler clocks, synchrony-free **proper-velocity** $w \equiv dx/d\tau$ defined in the map-frame, and the frame-invariant **proper-acceleration** α experienced by the traveler, which for unidirectional motion in flat space-time equals $(1/\gamma)dw/d\tau$ where $dw/d\tau$ is the **bookkeeper-acceleration** *in proper units* linked to momentum-changes seen in the map frame. Acceleration from the traveler perspective where the causes of motion are felt is key, because as Galileo and Newton demonstrated in the 17th century, those causes of motion are intimately connected to this second-derivative of position as a function of time.

The relationships above allow us to write proper-acceleration as the proper-time derivative of hyperbolic velocity-angle or **rapidity** η, defined by setting $c\sinh[\eta]$ equal to proper-velocity w in the acceleration direction. These relationships in turn simplify at low speeds (as long as we can also treat space-time as flat) as shown at right below, because one can then approximate the proper-values for velocity and acceleration (Fig. 2) with coordinate-values $v \equiv dx/dt$ and $a \equiv dv/dt$.

$$\begin{array}{rlcll} \text{proper-vel.} & c\sinh[\eta] = \frac{dx}{d\tau} & \rightarrow & v = \frac{dx}{dt} & \text{coord.-vel.} \\ \text{proper-accel.} & \alpha = c\frac{d\eta}{d\tau} & & a = \frac{dv}{dt} & \text{coord.-accel.} \end{array} \tag{1}$$

As mentioned above, treating space-time as flat requires that our map frame be seen as a free-float-frame (i.e. one experiencing no net forces). Introductory courses therefore concentrate on drawing out uses for the kinematic equations on the right hand side above. We provide the ones on the left, to show that only a bit of added complication will allow one to work in curved spacetimes and accelerated frames where **geometric-accelerations** as well as force-related proper-accelerations have an impact on the bookkeeper-accelerations observed.

IV. SAMPLE EXTENSIONS

The few-paragraph introduction to the "Galilean kinematic approximation", outlined in the teaser above, is meant to serve as a self-sufficient lead-in on which no class time at all need be spent. (Of course it might not hurt to mention that time t in the text might by default refer to time on clocks connected to one's map-yardsticks.) The following extensions are therefore provided only to illustrate how the distinctions made in that introduction are "relativity smart".

The example problems might also help empower faculty/students in exploration of questions not covered in

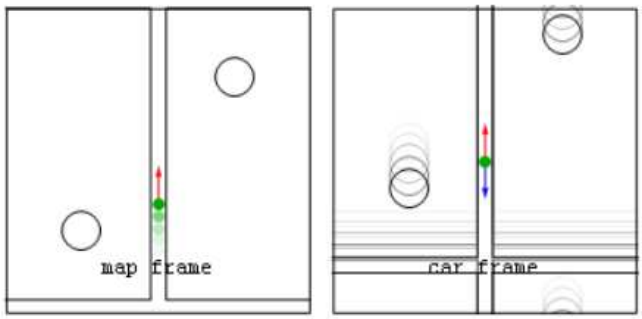


FIG. 3: Proper (red) and geometric (blue) accelerations on leaving a stop-sign.

standard courses but within range of available math e.g. to the extent that time and motivation outside of class is available. Experimental physics education research, of course, is still needed to uncover the problems that faculty and students at various levels will encounter by such explorations. This in turn, will be important to text authors downstream to determine how minor changes like this might help to constructively evolve the standard course.

A. everyday problems

We first look at problems that a traveler-kinematics introduction might help one address in the relatively low-speed activities characteristic of everyday life on earth.

1 accelerated frames

From the vantage point of any reference frame (accelerated or not) in any spacetime (flat and/or curved), general relativity tells us in four-vector terms[24] that momentum & energy changes per unit mass i.e. **4-vector bookkeeper-accelerations** $dU^\lambda/d\tau$ arise directly from some combination of (i) **4-vector proper-accelerations** A^λ caused by forces acting on a traveling object and (ii) frame-dependent **4-vector geometric-accelerations** $-\Gamma^\lambda_{\mu\nu}U^\mu U^\nu$ that act on every ounce of that object's being.

From the vantage point only of unaccelerated or "inertial" reference frames, Newton's laws further tell us that three-vector momentum changes per unit mass i.e. coordinate-accelerations $d\vec{v}/dt$ of objects traveling at speeds small compared to "lightspeed" are more simply caused by the net-force $\sum F$ per unit mass acting on that object, provided that we also think of gravity as a force itself (albeit one which acts on every ounce of our object's being).

Beyond this, general relativity's equivalence principle[19,25] gives us the good news that Newton's second law works well locally at low speeds whether or not your frame is accelerated, as long as geometric-accelerations other than gravity may also be treated as proper-accelerations due to forces that act on every ounce of an object's being. In other words, questions about inertial forces in an accelerated frame, e.g. of your car when you step on the gas or take a curve at constant speed, can be treated in this context.

Example Problem IV.A.1a: You are passenger in a car accelerating at 0.5 gee from rest at a stop sign. If your bookkeeper-acceleration (in the accelerated frame of your car) is zero, then what backwards geometric-acceleration on every ounce of your being appears (in that accelerated frame) to be canceling out the forward proper-acceleration caused by the horizontal force on you from the back of your seat?

Example Problem IV.A.1b: You are in a car going at constant speed v around a leftward curve of radius R. In order to hold your bookkeeper-acceleration (in the accelerated frame of your car) to zero, what rightward geometric-acceleration on every ounce of your being must appear (in that accelerated frame) to cancel out the centripetal proper-acceleration caused by the leftward force of your car seat on you? Also in which direction would the resulting torque appear to operate, if your center of mass is located above your point of contact with the seat?

Example Problem IV.A.1c: You are pilot of an inverted aircraft going 200 mph at the top of a half-mile radius loop-the-loop. What net geometric-acceleration toward the cupholder is the coffee feeling in your now-inverted cup?

In most books, of course, only gravity's geometric-acceleration will be treated as due to a force that acts on every ounce of an object. Other local effects of geometric-acceleration are avoided by applying Newton's Laws only in inertial frames that are not experiencing motion-related geometric-acceleration.

For shell frame observers experiencing gravitational acceleration g, we might amend Newton's 2nd law for a frame undergoing longitudinal (tangential) and/or transverse (radial) acceleration $\vec{a}_{\text{frame}}$ under non-relativistic conditions to read something like:

$$\sum_{i=1}^{N} \vec{F}_i + m\vec{g} - m\vec{a}_{\text{frame}} \simeq \frac{d\vec{v}}{dt} \tag{2}$$

Here the proper (force-related) and geometric (kinematic) accelerations are on the left side of the equation, while the bookkeeper-acceleration (e.g. of a piece of ice on the floor of an accelerated vehicle) is on the right. As you can see, the effect of gravity $m\vec{g}$ enters this equation in the same way whether it is conceptualized as the result of a force, or the result of a geometric acceleration that acts on every ounce of an object's being.

Thus introducing gravity as an effect due to spacetime curvature, that may be locally-approximated as a $1/r^2$ force, gives students heads-up at the outset that the differential passage of time is involved, that applied science is all about strategic approximation, and that Newton's Laws may with caution prove useful even in accelerated

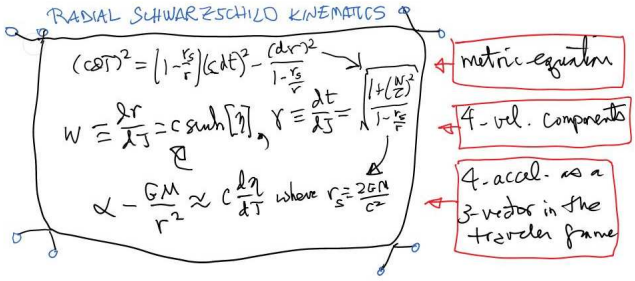


FIG. 4: Traveler-kinematics in a Schwarzschild shell-frame.

frames.

2 *gravitational time dilation*

The traveler kinematic generalizes nicely to curved spacetime as well as accelerated frames, as shown in the (1+1)D Schwarzschild metric example in Fig. 4. As in the accelerated-frame case, more care must be taken than in the Galilean-approximation since geometric-accelerations of only local utility are likely to be present here as well.

In particular the stationary (i.e. zero proper-velocity) Lorentz-factor in the far-time Schwarzschild metric, namely $dt/d\tau = 1/\sqrt{1-2GM/(c^2r)} \simeq 1+(1/c^2)GM/r$, is directly linked to the gravitational potential-energy per unit mass (GM/r) that we need to escape the planet. As discussed later, such differential-aging effects also occur for accelerated-frames in flat space-time.

Thus when a dropped object accelerates downward it is confirmation that objects closer to the earth's center are aging more slowly than objects further away. This differential-aging must be taken directly into account when estimating your location with a global positioning system, because GPS satellites operate at altitudes where the effect of gravitational time-dilation is significant.

Example Problem IV.A.2a: If you spend one quarter of your first decade standing up as a 1.8 meter tall adult, and very little of that time standing on your head to cancel the effect, about how much older is your head than your feet?

Example Problem IV.A.2b: If the official formation-date for the earth was 4.5 billion years ago, how much older is the earth's surface, than its center, today?

Note that these problems involve small differences between relatively large numbers, so they might have to be done with very high precision calculations. For example, clocks 1.8 meters above the earth's surface only pick up about 1.96 additional attoseconds per second of elapsed time at the earth's surface.

The *moving* Schwarzschild Lorentz-factor $dt/d\tau = Sqrt[1+(w/c)^2]/Sqrt[1-2GM/(c^2r)]$, which is a product of stationary and motion-related terms, can of course also be used by students to examine time-dilation effects

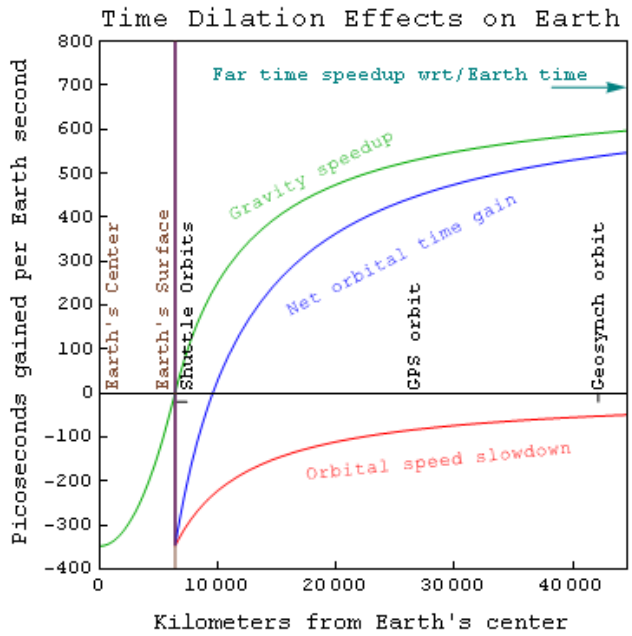


FIG. 5: Time-dilation effects in satellite global-positioning.

associated with satellite orbits. One must of course consider time at two radii (e.g. on earth and in orbit) instead of Schwarzschild far-time t per se, in which case one finds that low-earth orbit effects are dominated by satellite motion while high earth-orbit effects are dominated by gravitational time-dilation, as shown in Fig. 5.

B. anyspeed problems

In addition to working in accelerated frames and curved spacetime, the traveler-kinematic approach works regardless of speed. As a result, the short introduction above opens to the door to calculations involving velocity and acceleration at super as well as at sub relativistic speeds i.e. for proper velocities (and momenta per unit mass) well above **and** well below 1 lightyear per traveler year. Some examples of this are provided here.

1 *colliders vs. accelerators*

Although $v \ll c$, $w \ll c$ and $K \ll mc^2$ are all natural inequalities that define the sub-relativistic regime, where for example $\delta t \simeq \delta t$, coordinate-velocity v inequalities are not useful compared to the proper-velocity and kinetic-energy inequalities $w \gg c$ and $K \gg mc^2$ for defining a supra-relativistic regime, where for example $K \simeq pc$. Moreover, a proper-velocity of just one lightyear per traveler year i.e. $w \simeq c$ is a natural dividing point between those two limiting cases.

Relativistic-particle land-speed records also become more interesting in lightyears per traveler year (w in

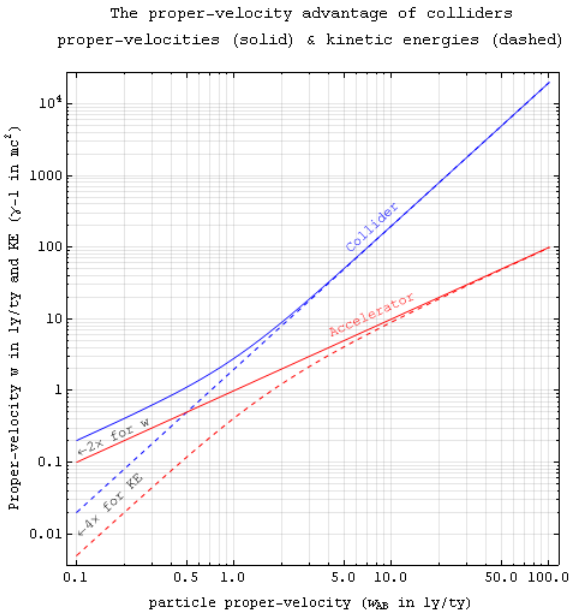


FIG. 6: The proper-velocity advantage of colliders due to Lorentz-factor multiplication.

[ly/ty]), than in lightyears per map year (v in [ly/y]). For example, a 45 GeV electron accelerated in 1989 by the Large Electron-Positron Collider (LEP) at Cern would have had a coordinate-speed v of only about sixty four trillionths shy of 1 [ly/y]. On the other hand, its proper-speed would have been around 88,000 [ly/ty], a number much more useful for comparisons from one run to the next.

To see this, note that proper-velocity $\vec{w} \equiv d\vec{x}/d\tau = \gamma v$ can be written as Lorentz-factor $\gamma \equiv dt/d\tau$ times coordinate-velocity $\vec{v} \equiv d\vec{x}/dt$. For unidirectional motion, proper-velocities simply add by the symmetric relation:

$$w_{AC} \equiv \gamma_{AC} v_{AC} = \gamma_{AB}\gamma_{BC}(v_{AB} + v_{BC}) \qquad (3)$$

i.e. the coordinate-velocities add while the Lorentz-factors multiply. Reporting accelerator speeds in proper units (e.g. of 1 [ly/ty]) better connects high-speed numbers to properties (like momentum and energy) familiar at low speeds, and it makes the numbers more impressive as well. It also illustrates the collider advantage much more clearly.

Example Problem IV.B.1a: What's the proper land-speed-record for protons accelerated in the laboratory on earth? How about for electrons, or for iron nuclei?

Example Problem IV.B.1b: If two of the protons mentioned in the previous problems were put onto a head-on collision-course, what would be the *relative* proper-speed of collision, obtained at likely less than twice the cost of running one of those protons into a stationary target?

Voters enjoying land-speed records in [ly/ty] might really enjoy hearing about the collider advantage. As see in Fig. 6, accelerating two 45 GeV electrons and colliding them takes the proper land-speed for a single electron of 88,000 [ly/ty] up to a relative collider speed of $w_{AC} \simeq 88,000^2(1+1) \sim 1.55 \times 10^{10}$ [ly/ty]. This is quite a bargain over one accelerator, for something like twice the cost.

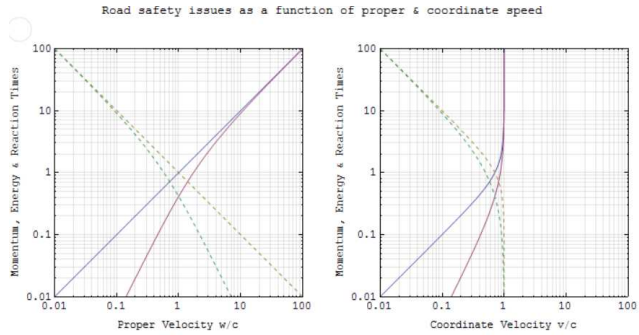


FIG. 7: Momentum (blue), energy (red) plus driver (yellow) and pedestrian (green) reaction times as a function of proper (left) or coordinate (right) speed.

Of course proper speed is not the speed experienced by the couch potato watching the particle go by, but it is also perhaps **the** measure of speed which is least dependent on the existence of a set of synchronized bookkeeper or map-clocks. Given the local nature of time, as discussed below, this may be a non-trivial advantage.

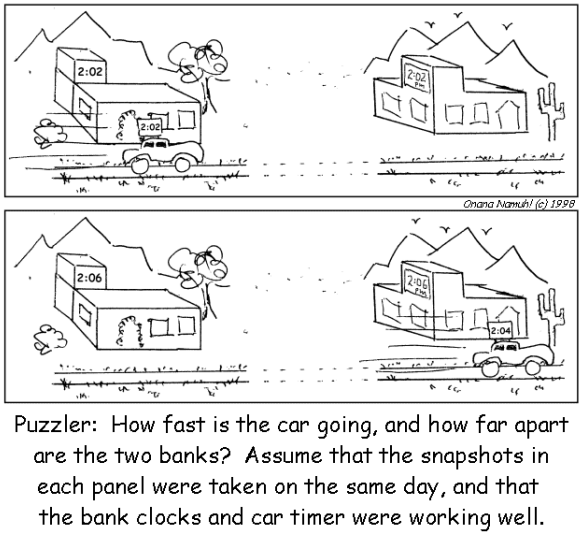
2 *anyspeed road safety*

Unlike coordinate-velocity dx/dt, proper-velocity is proportional to momentum at any speed, has no upper limit, and only makes local assertions about the reading on one clock (the traveler clock). One might however wonder about its relative utility. At high speeds (or in a dream-world of George Gamow's Mr. Tompkins[26] where lightspeed was 55 miles per hour) would highway speed-limits be more usefully expressed in proper or coordinate velocity units?

Example Problem IV.B.2a: On a car trip with 300 map-miles remaining, is it proper-velocity or coordinate-velocity which determines most directly how much longer you have to remain in the vehicle? Is it proper-velocity or coordinate-velocity which determines most directly what time you can be expected to arrive on map-clocks?

Example Problem IV.B.2b: What upper limit on vehicle momentum, vehicle kinetic energy, driver reaction-time for obstacle avoidance, and pedestrian reaction-time for oncoming-vehicle avoidance, is provided by a proper-speed limit of w/c in the range of positive real numbers? How would these relationships change for a coordinate-speed limit of v/c instead?

Example Problem IV.B.2c: Your 1000 kg space-roadster accelerates from rest to 0.707 lightspeed i.e.



Puzzler: How fast is the car going, and how far apart are the two banks? Assume that the snapshots in each panel were taken on the same day, and that the bank clocks and car timer were working well.

FIG. 8: Single-frame time-dilation puzzler.

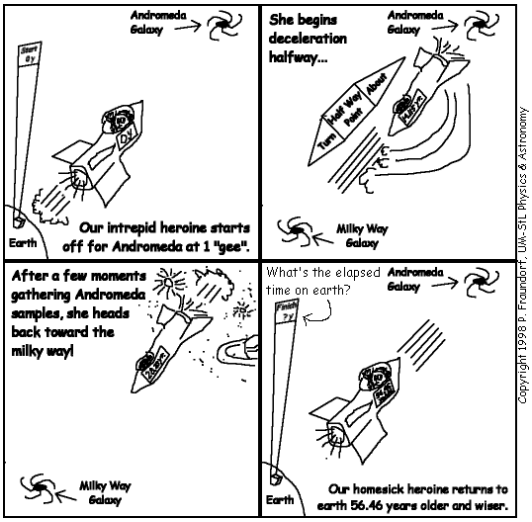


FIG. 9: An accelerated roundtrip puzzler.

0.707 [lightseconds/map-second]. The kinetic energy acquired on this "jump to lightspeed" is equivalent to how many gallons of gasoline, each of which contains 100 million Joules of energy available to do work?

3 single-frame time-dilation

As long as only one map-frame is required, solving time-dilation problems may be much easier using proper-velocity in the traveler-kinematic, than using coordinate-velocity, since the former relates directly to the relation between map-distance covered in a given amount of traveler-time:

Example Problem IV.B.3a: Muons with a lifetime of $\Delta\tau = 50$ [micro-seconds] are created at $\Delta x = 100$ [km] altitude, in collisions between air molecules and high-energy cosmic rays. Since these muons routinely make it to sea level for detection before their on-board clocks cause them to decay, how fast (earth distance traveled per unit earth time) must they be moving?

Example Problem IV.B.3b: The timer carried by a high speed airtrack glider registers only one third of the elapsed time recorded by gate clocks, on the trip between a pair of synchronized gates fixed to the airtrack. How fast was the glider traveling?

Example Problem IV.B.3c: The highly-accurate timer carried by a fast-moving vehicle registers only 99 percent of the elapsed-time recorded by synchronized bank clocks that it passes. How fast was the vehicle traveling?

The caveat is that length-contraction always requires two frames of synchronized clocks, so careful treatment of length-contraction effects should likely await the introduction of Lorentz transforms. We do attempt to address some questions of extended radar-time simultaneity without them,later on in this paper.

4 accelerated roundtrips

Given that acceleration is not always discussed much in special-relativity texts[30], one might imagine that equations for any-speed acceleration are irrelevant to everyday life. On the contrary thanks to proper-acceleration's frame-invariance and general relativity's equivalence-principle, which allows Newton's laws to work locally in accelerated (non-free-float) frames with help from non-proper geometric (affine-connection) forces that act on every ounce of an object's being, proper-acceleration allows one to explain the difference between gravity and most other intro-physics forces[22] from the first day of class.

Thus the local metric equation allows one to address acceleration problems, as well as those involving constant velocity. These may be quite relevant, for example, to the question of long-distance space-travel. Since humans are not adapted to accelerations above 1-gee but can take advantage of a 1-gee acceleration to both minimize travel-time and keep our bodies in shape, let's focus specifically on 1-gee constant proper-acceleration roundtrips as the ideal if we can find a way to deal with the problems of fuel mass and high-speed collisions.

A simple model for "one-gee" round-trips might be someone doing jumping-jacks. Technically of course, on earth at least, these take place in a non-free-float frame in which launch is accomplished with help of a proper-acceleration while the return trip is accomplished with help of a geometric-force that acts on every ounce of one's being.

The good news for interstellar roundtrips, if you simply put in the numbers, is that how far one can go in a given amount of elapsed traveler-time exceeds the distance one

can go in a Galilean world (without a finite value for space-time constant c) by a ridiculous amount. In other words, relativity opens up rather than closes down possibilities for interstellar travel in terms of time-elapsed on traveler-clocks[31]. It's the couch-potatoes at home that relativity hurts, not the travelers themselves.

For instance, a 57-year 1-gee roundtrip using the low-speed (non-relativistic) equations for constant coordinate-acceleration above would at most allow one to go about 200 lightyears and back. The same 57-year trip using the relativistic equations for constant proper-acceleration would take you all the way to Andromeda galaxy 2 million lightyears away and back.

Example problem IV.B.4a: How much traveler-time would elapse on a 1-gee constant proper-acceleration roundtrip to and from Sirius if it were 8.6 lightyears away from Earth? How much map time would elapse on the same trip? How much on-board fuel would be required at the start of each one-way leg of the trip, assuming that the fuel could be efficiently converted to photons directed in the desired thrust direction.

Example problem IV.B.4b: An enemy spaceship with its FTL-drive disabled drops out of hyperspace in the neighborhood of a ringworld habitat, traveling at 1 ly/ty radially away from the ringworld's star. A starfleet battle-cruiser capable of continuous 1 gee acceleration at rest nearby takes up the chase. How much cruiser time and enemy time elapses before the cruiser can catch up to the enemy? What are the ringworld coordinates and time for that encounter? What is the relative proper-speed of the two ships when that encounter occurs?

Example problem IV.B.4c: What's the traveler time-elapsed on a 1-gee constant proper-acceleration round-trip to Andromeda galaxy $2,538,000$ lightyears away?

The bad news is that carrying on-board fuel (even one-way) on these trips will make trips just to nearby stars difficult, and the thrust-profile for constant proper-acceleration very heavily front-loaded[22]. Extended times at 1-gee acceleration of course might make collisions with dust particles (or even hydrogen atoms) at ambient speeds a non-trivial problem as well.

5 *adding 3-vector velocities*

The traveler-kinematic also extrapolates nicely into a description of (3+1)D accelerated motion, as illustrated in Fig. 10. We've already discussed some advantages of unidirectional proper-velocity addition, in comparison e.g. to the addition of relative coordinate-velocities at high speed. We here explore the familiar "tail-to-nose" addtion of three-vector velocities, and in particular application of the dreaded "relative-velocity equation" $\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$.

Coordinate-velocity 3-vectors only add at speeds small compared to that of light. However, regardless of speed

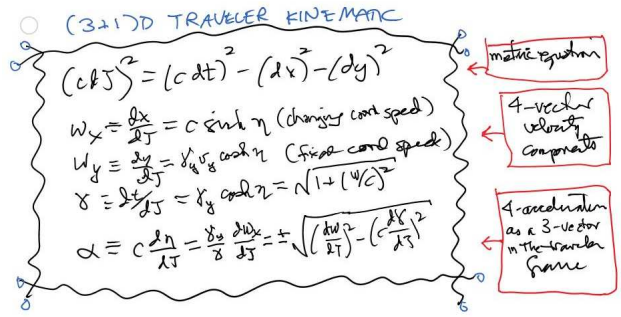


FIG. 10: The synchrony-free traveler-kinematic in (3+1)D.

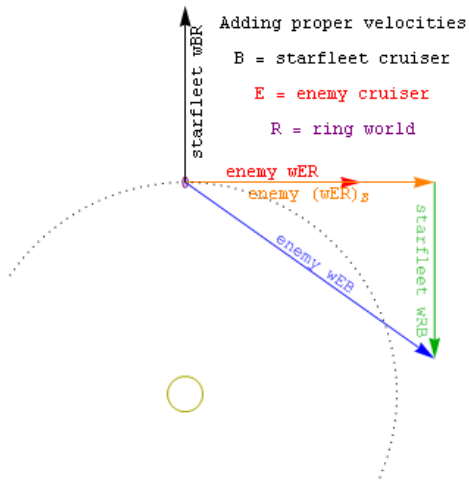


FIG. 11: Tail to nose proper-velocity addition.

local proper-velocities $\vec{w} \equiv d\vec{x}/dt$ do add naturally as 3-vectors, provided that we rescale the magnitude of the any "out-of-frame" vectors. As a result the familiar three-vector relative-velocity equation from introductory physics continues to work for proper-velocities regardless of speed, i.e.

$$\vec{w}_{\rm AC} = (\vec{w}_{\rm AB})_{\rm C} + \vec{w}_{\rm BC} \tag{4}$$

where Lorentz-factor $\gamma \equiv dt/d\tau = \sqrt{1+(w/c)^2}$, with the caveat that the magnitude-only of $w_{\rm AB}$ is rescaled into frame C using:

$$(\vec{w}_{\rm AB})_{\rm C} \equiv \left(\frac{\vec{w}_{\rm AB}\cdot\vec{w}_{\rm BC}}{(1+\gamma_{\rm AB})c^2} + \gamma_{\rm BC}\right)\vec{w}_{\rm AB} \tag{5}$$

.

This makes it possible to use familiar tools for addressing problems like the following:

Example Problem IV.B.5a: A starfleet battle-cruiser drops out of hyperspace in the orbital plane of a ringworld, traveling at 1 [ly/ty] radially away from the ringworld's star. An enemy cruiser drops out of hyperspace nearby at the same time, traveling 1 [ly/ty] in the rotation-direction of the ringworld's orbit, and in a direction perpendicular to the starfleet cruiser's radial-trajectory. What is the proper-velocity (magnitude and

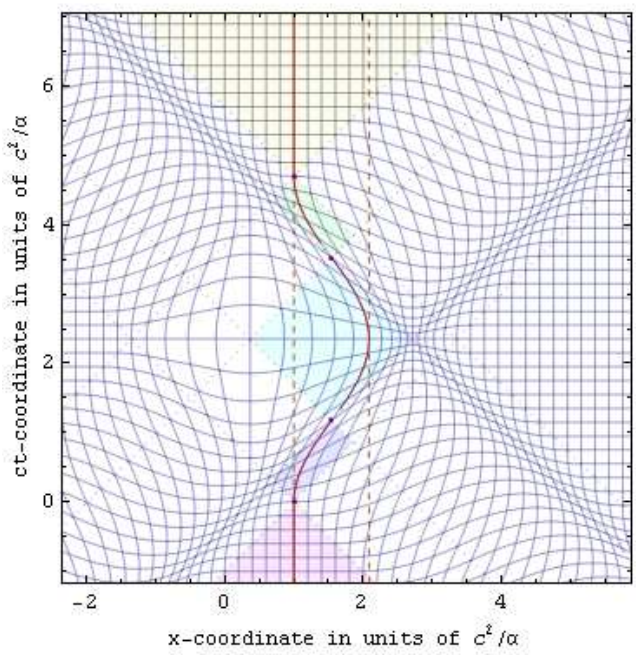


FIG. 12: In special-relativity with radar-time simultaneity[27], acceleration curves flat spacetime.

direction) of the enemy cruiser relative to the starfleet ship?

6 *acceleration-related aging*

Although beyond the scope of an introductory physics class, the curvature of spacetime coordinates for an accelerated frame in flat-spacetime can be visualized via that accelerated traveler's radar-time simultaneity $\rho - c\tau$ isocontours, as shown Fig. 12. This allows one to examine differential-aging from the traveler's perspective, associated with event-points on arbitrary world lines in this space.

The map-frame ct versus x radar-time simultaneity plot in the figure at right shows how acceleration, in this case of a 1-gee proper-acceleration round-trip lasting 4 traveler-years, distorts distances (blue vertical mesh-lines) and simultaneity (blue horizontal mesh-lines) experienced by that accelerated observer. For objects that are extended along the line of their acceleration, these distortions in space and time will occur even across an accelerated-objects own length.

For example, in addition to the metric-equations motion-related time-dilation in which $\delta t_{\text{traveler}}/\delta t_{\text{map}} = \sqrt{1-(v/c)^2}$, for accelerated objects of length L in the direction of proper-acceleration α, one finds an acceleration-related time-dilation of the form: $\delta_{\text{trailing}}/\delta t_{\text{leading}} = e^{-\alpha L/c^2}$. Here the leading-edge of the object is in the direction of the acceleration α, not necessarily in the direction of travel.

Example Problem IV.B.6a: Imagine that standing on the earth's surface at radius R is like undergoing constant proper-acceleration for an extended period of time. What estimate would this give for the differential aging between your head and your feet as a 1.8 meter tall adult, if you spent 25 percent of your life standing during the last 10 years?

This calculation, using Dolby and Gull's radar-time metric, gives essentially the same differential-aging as does the Schwarzschild metric for this "stationary-acceleration" problem, with a simpler approximation. In fact, in this case one gets simply an increased tick rate of 109 attoseconds per second for each meter of height. This of course is only a local calculation, in comparison to the Schwarzschild one, since the acceleration due to gravity falls off with height h above the surface unless $h \ll R$.

However this sort of calculation is not very useful for spaceship problems. That's because a "rigid spaceship" at 1-gee acceleration will to first order force its leading and trailing portions into a Rindler coordinate system, whose length and clock rates will remain constant although trailing parts of the ship will in effect experience slightly greater proper-acceleration during acceleration-phase of the trip.

For the 1-gee proper-acceleration of a standing human in the vertical direction, this differential-aging between head and foot on the order of $1 - 2 \times 10^{-16}$. This means that if you stand up (or sit tall) for a sizeable fraction of your lifetime, your head may be a few-hundred nanoseconds older than your feet. This is a small effect for humans, but as discussed earlier its quite significant for global-positioning satellites for which nanosecond timing-errors give rise to macroscopic errors in position.

The differential-aging of accelerated objects is linked to the gravitational time-dilation experienced by shell-frame objects in a planet's gravitational field, because a proper-acceleration is needed to keep objects in such a field from following a rain-frame trajectory.

V. CONCLUSIONS

To recap, we suggest here a few paragraphs of introduction about: (i) time as a local variable for describing motion regardless of speed using a traveler-kinematic involving only frame-invariant or synchrony-free variables, and (ii) the Galilean-kinematic approximation, which is the subject of most courses because it works so well for engineering problems on earth. We further provide example problems for applying the traveler-kinematic which are mathematically accessible to most introductory physics students, should they be tempted to explore them further outside the bounds of a traditional class. Appendices are provided as background on the metric equations, Lorentz-factor calculations, and constant acceleration integrals that underlie these well-known results.

Concerning wider applications for the traveler kinematic, four-vectors are of course written in the traveler-kinematic i.e. in terms of derivatives with respect to proper-time. Moreover the *free-particle* Lagrangian in curved space-time, when parameterized in terms of proper-time, is simply $-mc^2$. This yields the most elegant and comprehensive prediction of free-particle motion yet: In the absence of proper-acceleration, objects move so as to maximize aging[28,29] i.e. elapsed proper-time.

More generally of course the metric-equation strategy used above for relating proper-acceleration to bookkeeper acceleration works in any curved space-time or accelerated frame. Of course relation between the two involves a covariant-derivative with connection-coefficients, whose multi-component sums are beyond the interest of most undergraduates.

Nonetheless, a closer look at everyday curvatures (like the gravity around a planet) may be of interest to some. Questions to explore there, for instance, might include: (a) how satellite orbits can be predicted by choosing the path of maximal aging, or (b) how the Schwarschild-metric yields within one expression the equations for both shell-frame gravity and accelerated-frame centrifugal-force.

ACKNOWLEDGMENTS

I would like to thank Roger Hill, Bill Shurcliff, and Edwin Taylor for their enthusiasm about new ways to look at old things.

APPENDIX A: FINDING $dt/d\tau$ FROM THE METRIC

Here we outline the basis relationships using physical units for the benefit of folks interested in some concrete calculations.

The metric equations discussed here are the Minkowski metric, the Schwarzschild metric, and the flat-space radar-time constant proper-acceleration metric. The Lorentz factor defined as $dt/d\tau$ for these is simply a rearrangement of the metric equation itself: Divide through by $c\delta\tau$, and solve for $dt/d\tau$ by moving terms around and taking the square root.

For the (1+1)D Minkowski (flat-space) metric

$$(c\delta\tau)^2 = (c\delta t)^2 - (\delta x)^2 \tag{A1}$$

, this yields the special-relativistic Lorentz factor $\gamma \equiv dt/d\tau = \sqrt{1+(w/c)^2} = 1/\sqrt{1-(v/c)^2}$ in terms of proper-velocity $w \equiv dx/d\tau$ or coordinate-velocity $v \equiv dx/dt$.

For the (1+1)D Schwarzschild (gravitational) metric with only radial motion, we start with:

$$(c\delta\tau)^2 = (1-\frac{r_o}{r_{\rm far}})(c\delta t_{\rm far})^2 - \frac{(\delta r_{\rm far})^2}{(1-\frac{r_o}{r_{\rm far}})}. \tag{A2}$$

On earth's surface the metric equation doesn't differ much from the Minkowski case since $r_o/r_{\rm far} \simeq 1.39117 \times 10^{-9}$, given that **event-horizon radius** $r_o = 2GM/c^2 \simeq 8.87$ [mm], where G is the universal gravitation constant and M the earth's mass. Nonetheless, the effects of gravity are far from negligible!

The Lorentz-factor in this case becomes the product of a motion-related and a radius-dependent term, namely $\gamma \equiv dt_{\rm far}/d\tau = \sqrt{1+(w/c)^2}/\sqrt{1-r_o/r_{\rm far}}$, where bookkeeper coordinates $t_{\rm far}$ and $r_{\rm far}$ represent "far-coordinates" i.e. values determined using the synchronized clocks and yardsticks of observers in flat space far away from our gravitational object.

Finally the (1+1)D accelerated-frame radar-time metric, by comparison, looks like

$$(cd\tau)^2 = e^{2\frac{\alpha x_\alpha}{c^2}}(c^2\delta t_\alpha^2 - \delta x_\alpha^2) \tag{A3}$$

. We will only be considering objects which are a fixed radar-time distance x_α from our accelerated traveler (and hence in this specific sense "stationary"), the Lorentz factor of interest is $\gamma \equiv \delta t_\alpha/d\tau = e^{-2x_\alpha\alpha/c^2}$.

APPENDIX B: FINDING α FROM $dt/d\tau$

Finding a relation between the frame-invariant magnitude of a travelers proper-acceleration α and parameters like velocity and position with help from the Lorentz-factor calculated from the metric is relatively easy with the Minkowski (flat-space) metric. This is because proper-velocity and the Lorentz-factor make up components of the 4-vector velocity whose proper-time derivative is the 4-vector proper-acceleration.

The problem is more complicated in curved space-time and in accelerated frames because of need for connection (i.e. geometric acceleration) terms in the covariant derivative that describes the 4-vector proper-acceleration. Here we consider only two special cases: Low speed radial motion relative to the shell-frame of a gravitating sphere, and travelers whose radar-distance from an accelerating map-frame is fixed.

APPENDIX C: CONSTANT ACCELERATION INTEGRALS

For a change of pace from most texts, let's discuss the assumptions needed for a *computer* to derive the equations of constant acceleration. For equations that work at any speed, we'll also give you some practice treating time as a *local* instead of as a global variable i.e. as a value connected to readings on a specific clock.

1. low-speed results

If we define **coordinate-acceleration** $a \equiv dv/dt$ and **coordinate-velocity** $v \equiv dx/dt$ where x and t are **map-**

position and **map-time**, respectively, then holding constant the coordinate-acceleration a (which is not the acceleration felt by our traveler at high speeds) allows one to derive the $v \ll c$ low-speed constant coordinate-acceleration equations familiar from intro-physics texts for coordinate-velocity $v[t]$ and map-position $x[t]$. Can **you** do it?

The following is what Mathematica needs to pull it off:

```
FullSimplify[DSolve[{
  v[t] == x'[t],
  a == v'[t],
  x[0] == xo,
  v[0] == vo
},
{x[t], v[t]},
t
]
].
```

Here we've added the intial ($t = 0$) boundary-conditions by defining x_o as initial map-position and v_o as initial coordinate-velocity to eliminate the two constants of integration. Mathematica's output is:

```
{ {
  v[t] -> a t + vo,
  x[t] -> (a t^2)/2 + t vo + xo
} } .
```

Thus the equations for constant coordinate-acceleration in one direction may be written:

$$v = v_o + at \tag{C1}$$

$$x = x_o + v_o t + \frac{1}{2}at^2 = x_o + \frac{v^2 - v_o^2}{2a}, \tag{C2}$$

where as usual $\Delta f \equiv f_{\text{final}} - f_{\text{initial}}$ for any time-varying quantity f. Here the first equation tells us how coordinate-velocity v changes with elapsed map-time t, while the second tells us how map-position x changes with map-time t as well as with state-of-motion (the work-energy equation) since work is $W \simeq ma\Delta x$ and kinetic energy is $K \simeq \frac{1}{2}mv^2$.

In terms of increments instead of differentials for constant unidirectional acceleration, we can therefore also write: $a = \Delta v/\Delta t = \frac{1}{2}\Delta[v^2]/\Delta x$.

2. any-speed results

For equations that work at any speed, we begin by treating the "proper" time τ on the clocks of a traveler as a local-variable, whose value we'd like to figure out relative to the local value of the traveler's position x and time t on the yardsticks and synchronized clocks of a reference map-frame. The space-time Pythagorean theorem or "metric-equation" for flat space-time, namely $(c\delta\tau)^2 = (c\delta t)^2 - (\delta x^2)$ with "lightspeed" constant c, allows us to define minimally frame-variant "proper" values for the velocity and acceleration, as well as for the time, *experienced by our traveler*.

In particular **proper-velocity** w (map-distance x per unit **proper-time** τ) is just $w \equiv dx/d\tau \equiv c\sinh[\eta]$, where η is referred to as hyperbolic velocity-angle or **rapidity**. The frame-invariant **proper-acceleration** α felt by a traveler equals this "length-contracted" proper-velocity derivative $(1/\gamma)d^2x/d\tau^2 = cd\eta/dt$ i.e. constant c times the traveler-time τ derivative of rapidity η. Holding α fixed thus allows one to derive constant proper-acceleration equations that work at any speed for map-position $x[\tau]$ and proper-velocity $w[\tau]$.

In the above discussion we are using proper-time τ *local to the traveler's clocks* as the independent variable, so as to avoid thinking of time as a global variable. Note that unlike coordinate-velocity $v \equiv dx/dt$, proper-velocity $w \equiv dx/d\tau$ always equals momentum per unit mass and has no upper limit. Can **you** figure how map-position x and proper-velocity $w = c\sinh[\eta]$ depend on traveler-time τ, given this information?

The following is what Mathematica needs to pull it off:

```
FullSimplify[DSolve[{
  c Sinh[eta[tau]] == x'[tau],
  alpha == c eta'[tau],
  x[0] == xo,
  eta[0] == etao
},
{x[tau], eta[tau]},
{tau}
]
].
```

As before we specify two initial ($\tau = 0$) conditions, in this case for initial map-position x_o and initial rapidity or hyperbolic velocity-angle η_o. The result is:

```
{ {
  eta[tau] -> etao + (alpha tau)/c,
  x[tau] -> (alpha xo + c^2 (-Cosh[etao] +
    Cosh[etao + (alpha tau)/c]))/alpha
} } .
```

Note that we also get these bonus relationships: Traveler-speed on the map can be expressed in several ways, including: **Lorentz-factor** $\gamma \equiv dt/d\tau = \cosh[\eta] = \sqrt{1+(w/c)^2} = 1/\sqrt{1-(v/c)^2}$, where coordinate-velocity $v \equiv w/\gamma = c\tanh[\eta]$. For incremental changes when proper-acceleration is constant and all motion is along that direction we can also write $\alpha = \Delta w/\Delta t = c\Delta\eta/\Delta\tau = c^2\Delta\gamma/\Delta x = \gamma^3 a$.

All of the foregoing **assertions are local to the traveler's position in the map-frame** of yardsticks and synchronized clocks. If we use those synchronized clocks to define simulaneity between separated events, the above

also tells us about **traveler motion from the perspective of stationary observers anywhere on the map**. Hence these equations are spectacular for exploring constant-acceleration round-trips between stars.

Thus the equations, analogous to the Newtonian ones, for unidirectional constant proper-acceleration at any speed might be written:

$$w = c\sinh\left[\frac{\alpha\tau}{c} + \eta_o\right] = w_o + \alpha\int_0^\tau \gamma[\tau']d\tau' = w_o + \alpha\Delta t, \tag{C3}$$

$$x = x_o + \frac{c^2}{\alpha}\left(\cosh\left[\frac{\alpha\tau}{c} + \eta_o\right] - \cosh\left[\eta_o\right]\right) = x_o + \frac{c^2}{\alpha}\Delta\gamma, \tag{C4}$$

where again $\Delta f \equiv f_{\text{final}} - f_{\text{initial}}$, and $\eta_o \equiv \sinh^{-1}[w_o/c]$. Here the first equation tells us how proper-velocity w changes with elapsed traveler-time τ and map-time Δt, while the second tells us how map-position x changes with traveler-time τ as well as with state-of-motion (the work-energy equation) since work is $m\alpha\Delta x$ and change-in kinetic energy is $\Delta K = mc^2\Delta\gamma$ given that $K = (\gamma-1)mc^2$. These results generalize nicely to the (3+1)D case (Fig. 10) and to curved spacetime (Fig. 4), although more care must be taken than in the Galilean approximation since 3-vector magnitudes show more dependence on observer frame.

At low speeds ($w \ll c$) of course, map and traveler clock times go at the same rate i.e. $dt \simeq d\tau$, the velocity-parameters (coordinate, proper and angle) are essentially the same i.e. $v \simeq w \simeq c\eta$, coordinate and proper acceleration are about equal i.e. $a \simeq \alpha$, and the any-speed equations reduce to the low-speed ones discussed above.

APPENDIX D: COOL ANY-SPEED APPLICATIONS

Invariant proper-time τ is already finding its way into intro-physics and special-relativity books, e.g. to recognize that the frame a clock resides in is special when the topic of *time elapsed on that clock* comes up. Proper-velocity w and proper-acceleration α for a traveler are less consistently mentioned, but also have uses that may be of interest to introductory physics teachers. In this section we discuss a few of the possibilities.

1. proper-velocity at home

In the multi-directional case, moreover, a 3-vector equation very similar to the low-speed velocity equation can be written, namely $\vec{w}_{AC} = (\vec{w}_{AB})_C + \vec{w}_{BC}$. The only complication is that frame C's view of the out-of-frame proper-velocity $(\vec{w}_{AB})_C$ is in the direction of $\vec{w}_{AB} \equiv \gamma_{AB}\vec{v}_{AB}$ but must be rescaled in magnitude[22] before the addition will work.

Another fun, but less practical topic, is that of relativistic traffic safety which includes games in Mr. Tompkins style universes[26] where e.g. lightspeed is 55 mph. In such a universe, would interstate highways still need speed-limit signs? The answer is yes, and it would moreover be a proper and not a coordinate velocity limit.

To see this, simply consider (one at a time) which velocity-measure best reflects maximum possible collision-damage in terms of vehicle (i) momentum and (ii) kinetic energy, and which measure best reflects minimum chance for collision-avoidance in terms of (iii) driver and (iv) pedestrian reaction-time. At all speeds, both vehicle momentum and kinetic energy scale nicely with proper-velocity while their dependence on coordinate-velocity goes through the roof as $v \to c$. Similarly driver reaction-time decreases, as does pedestrian reaction-time after the warning photon arrives, in a complementary way with increasing proper-velocity but not with coordinate-velocity[22].

Thus in our $c = 55$ mph universe, limiting travelers to speeds of less than 55 map-miles *per traveler-hour* makes more sense than limiting them to less than 55 or even 54.991 map-miles *per map-hour*. Not only would raising the limit to 60 mph remain a viable option, but as an added bonus the speedometer-reading for proper-velocity divided into destination distance directly answers the question that kids in the back seat are asking i.e. "How long (to me) before we get there?"

APPENDIX E: CURVING SPACE-TIME

Our analysis in the first section, of time as local to the clocks used to measure it, was in part to distance ourselves as much as possible from a discussion of extended simultaneity. In this section, for dealing with accelerated travelers we choose a radar-time model for extended simultaneity[27] (instead of the tangent free-float-frame model) in order to show students how proper-acceleration curves space-time for the traveler all by itself. This model in hand, the door my open a bit wider to experimentation by interested students with simple gravitational metrics in the spirit of Taylor and Wheeler's "Exploring Black Holes" text[18], whose pre-publication draft-title was "Scouting Black Holes: Exploring General Relativity with Calculus" likely in part to inspire a closer look by intro-physics students.

In this note, we don't have the opportunity to develop the equations to treat curved space in detail. Instead, therefore, we focus on visualizations and on a few bottom-line relationships that might pique a student's interest.

1. acceleration-related curvature

The map-frame ct versus x radar-time simultaneity plot in Fig. 12 shows how acceleration, in this case of

a 1-gee proper-acceleration round-trip lasting 4 traveler-years, distorts distances (blue vertical mesh-lines) and simultaneity (blue horizontal mesh-lines) experienced by that accelerated observer. For objects that are extended along the line of their acceleration, these distortions in space and time will occur even across an accelerated-object's own length.

For example, in addition to the metric-equation's motion-related **time-dilation** in which:

$$\delta\tau_{\text{traveler}} = \delta t_{\text{map}}\sqrt{1-\left(\frac{v}{c}\right)^2}, \tag{E1}$$

for accelerated objects of length L in the direction of proper-acceleration α, one finds an acceleration-related time-dilation of the form:

$$\delta\tau_{\text{trailing}} \simeq \delta\tau_{\text{leading}}\sqrt{1-\frac{2\alpha L}{c^2}}. \tag{E2}$$

Here the leading-edge of the object is in the direction of the acceleration α, not necessarily in the direction of travel.

For the 1-gee proper-acceleration of a standing human in the vertical direction, this **differential-aging** between head and foot becomes $dt_{\text{foot}}/dt_{\text{head}} \simeq 1 - 2\times 10^{-16}$. This means that if you stand up (or sit tall) for a sizeable fraction of your lifetime, **your head may be a few-hundred nanoseconds older than your feet**. This is a small effect for humans, but as discussed below (and illustrated in Fig. 5) it's quite significant for global-positioning satellites for which nano-second timing-errors give rise to macroscopic errors in position.

2. gravity's acceleration

Einstein's general-relativity shows how a gravitational-acceleration that is the same for all masses can be seen to result from a mass-related static-distortion of space-time. This can be described most simply with a modified metric-equation of the form:

$$(c\delta\tau_{\text{radius-r}})^2 = (1-\frac{r_o}{r})(c\delta t_{\text{far}})^2 - \frac{(\delta x_{\text{far}})^2}{(1-\frac{r_o}{r})}. \tag{E3}$$

On earth's surface the metric equation doesn't change by much since $r_o/r \simeq 1.39117\times 10^{-9}$, given that **event-horizon radius** $r_o = 2GM/c^2$, where G is the universal gravitation constant and M the earth's mass.

Nonetheless, this modified-metric gives rise to **gravity's geometric-acceleration** of GM/r^2 that at earth's surface becomes $g \simeq 9.8$ meters per second-squared, which must be countered by an upward proper-force of mg (as shown later in this course) to keep a **shell-frame** observer's radius fixed in the neighborhood of a planet. That's because shell-frames (of fixed radius) are not free-float-frames. Around gravitational objects, free-float-frames are sometimes called **rain-frames** instead.

The space-time curvature associated with gravity's geometric-acceleration also distorts space and time. One result of this is the gravitational time-dilation of global-positioning-system (GPS) satellites, as well as of your head, relative to your boots on the ground.

As with the previous two expressions for differential-aging, this dilation is also linked to an expression involving $\sqrt{1-2\text{energy}/mc^2}$, namely:

$$\delta\tau_{\text{radius-r}} \simeq \delta t_{\text{far}}\sqrt{1-\frac{2GM}{rc^2}}, \tag{E4}$$

where potential-energy per unit mass at radius r (also to be shown later in the course) is GM/r. This further means that if clocks at the earth's center and surface began ticking together on the day when earth's formation from the solar-nebula was complete, since then **time-elapsed at earth's center is about a year less than on the surface**. Such differential-aging effects are even more severe with extremely dense objects, like **neutron stars** and the event-horizons of **black holes**.

REFERENCES

* pfraundorf@umsl.edu

[1] K. P. Burnham and D. R. Anderson, *Model selection and multimodel inference: A practical information-theoretic approach* (Springer Science, NY, 2002), 2nd ed.

[2] P. C. Gregory, *Bayesian logical data analysis for the physical sciences: A comparative approach with Mathematica support* (Cambridge U. Press, Cambridge UK, 2005).

[3] J. F. Reichert, *A modern introduction to mechanics* (Prentice Hall, Englewood Cliffs NJ, 1991).

[4] Moore, *Six ideas that shaped physics* (McGraw-Hill, NY, 2003), 2nd ed.

[5] L. A. Bloomfield, *How things work* (Wiley, New York NY, 2013), 5th ed.

[6] T. A. Moore and D. V. Schroeder, Am. J. Phys. **68**, 1159 (2000).

[7] P. Fraundorf, Am. J. Phys. **71**(11), 1142 (2003).

[8] C. Kittel and H. Kroemer, *Thermal Physics* (W. H. Freeman, New York, 1980), p. 39.

[9] K. Stowe, *Introduction to Statistical Mechanics and Thermodynamics* (Wiley & Sons, New York, 1984), p. 116.

[10] C. Garrod, *Statistical Mechanics and Thermodynamics* (Oxford University Press, New York, 1995).

[11] D. V. Schroeder, *An Introduction to Thermal Physics* (Addison-Wesley, San Francisco, 2000).

[12] E. F. Taylor, S. Vokos, J. M. O'Meara, and N. S. Thornber, Computers in Physics **12**, 190 (1998).

[13] J. R. Primack and N. E. Abrams, *The view from the center of the universe* (Riverhead Books, NY, 2006).

[14] R. E. Scherr, P. S. Schaffer, and S. Vokos, Phys. Educ. Res. - Am. J. Phys. Suppl. **69**(7), S24 (2001).

[15] R. E. Scherr, Am. J. Phys. **75**, 272 (2007).
[16] O. Levrini and A. A. diSessa, Phys. Rev. Special Topics: PER **4**, 010107 (2008).
[17] A. Pais, *Subtle is the Lord...* (Oxford University Press, 1982), cf. note about Einstein's evolving view of Minkowski's approach on page 152.
[18] E. Taylor and J. A. Wheeler, *Exploring black holes* (Addison Wesley Longman, 2001), 1st ed.
[19] J. B. Hartle, *Gravity* (Addison Wesley Longman, 2002).
[20] F. W. Sears and R. W. Brehme, *Introduction to the theory of relativity* (Addison-Wesley, NY, New York, 1968), section 7-3.
[21] W. A. Shurcliff, *Special relativity: The central ideas* (1996), 19 Appleton St, Cambridge MA 02138.
[22] P. Fraundorf, *Metric-1st and entropy-1st surprises*, arXiv:1106.4698 [physics.gen-ph] (2013), URL `http://arxiv.org/abs/1106.4698`.
[23] E. Taylor and J. A. Wheeler, *Spacetime physics* (W. H. Freeman, San Francisco, 1963), 1st ed., contains some material not found in the 2nd edition.
[24] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[25] T.-P. Cheng, *Relativity, gravitation and cosmology* (Ox, 2005), page 6: in special relativity... "we are still restricted to" ... "inertial frames of reference" and hence no acceleration.
[26] G. Gamow, *Mr. Tompkins in paperback* (Cambridge University Press, 1996), illustrated by the author and John Hookham.
[30] A. P. French, *Special relativity*, The M.I.T. Introductory Physics Series (W. W. Norton, New York, 1968), page 154: "...acceleration is a quantity of limited and questionable value in special relativity.".
[31] C. Lagoute and E. Davoust, Am. J. Phys. **63**, 221 (1995).
[27] C. E. Dolby and S. F. Gull, Amer. J. Phys. **69**, 1257 (2001).
[28] E. F. Taylor, Am. J. Phys. **71**, 423 (2003).
[29] T. A. Moore, Am. J. Phys. **72**(4), 522 (2004).