

Proper-velocity and $f \leq m\alpha$ from the metric

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By specifying the clocks used to track motion in introductory physics, one sets the stage for introduction of the proper-time τ elapsed on traveler-clocks via Minkowski's space-time version of Pythagoras' theorem. Taking derivatives of event-coordinates with respect to τ then lets one define 3-vectors for both proper-velocity $\vec{w} \equiv \gamma\vec{v}$ and proper-acceleration $\vec{\alpha}$ which *at all speeds* retain characteristics of their low-speed coordinate-analogs. Dynamical quantities emerge on multiplying these derivatives by rest-mass m , which also shows that scalar $f \equiv dp/dt$ is less than or equal to $m\alpha$ with equality in the unidirectional case. Thus by sticking with one "synchronous" frame, via the metric equation, students can develop skill with the equation underpinning magnetism and gravity from a starting point that's both familiar and concrete.

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I. INTRODUCTION

A. Map-based motion

This paper is not about new relationships, but about the synthesis of reasonably well-known relationships into a description of special relativity which is patterned after the description that introductory physics students are already familiar with from their classical physics courses. Enthusiasm for more concrete approaches like this is not new either.

Bell[1], Taylor & Wheeler[2, 3], Lagoute & Davost[4], Shurcliff[5], Moore[6], Dolby & Gull[7], Cook[8] and other experts in the use of spacetime's frame-invariants for education have encouraged introductions via the metric equation, and/or from the vantage point of a single map-frame of co-moving yardsticks and synchronized clocks. We expand on that challenge here by showing how introduction of one new variable (proper-time) via one added equation (the flat-space metric equation) sets the stage for a logical narrative, patterned after that used for Newtonian dynamics, that yields all of special relativity (plus useful steps toward electromagnetism and general relativity) from the vantage point of a single frame.

Key to this process is use of several less-familiar ideas. These include:

(i) explicit mention of frame-relevance for variables e.g. by use of Shurcliff's "eifo"-prefix (Effective for the Indicated Frame Only) for quantities whose value is important to observers in an arbitrary frame but command no all-frame respect.

(ii) proper-velocity $\vec{w} \equiv d\vec{r}/d\tau$, the distance per unit traveler-time[9], retains many of the properties that ordinary velocity loses at high speed.

(iii) Proper-acceleration $\vec{\alpha}$, experienced relative to a locally co-moving free-float-frame[2, 10], helps when accelerating, speeding, and in curvy space-time.

(iv) How some of the space-like effect of sideways "felt" forces moves into the reference-frame's time-domain at

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TABLE I: Cartesian 3-vectors at map-time $t \Rightarrow$ Newtonian dynamics.

component	displacement \vec{r}	velocity \vec{v}	acceleration \vec{a}	momentum \vec{p}	net-force $\Sigma\vec{F}$
X^μ	R^μ	$U^\mu \equiv \frac{dR^\mu}{dt}$	$A^\mu \equiv \frac{dU^\mu}{dt}$	$P^\mu \equiv mU^\mu$	$\Sigma F^\mu \equiv mA^\mu$
X^1	x	$\frac{dx}{dt} \equiv v_x$	$\frac{dv_x}{dt} \equiv a_x$	$mv_x \equiv p_x$	$\frac{dp_x}{dt} \equiv f_x$
X^2	y	$\frac{dy}{dt} \equiv v_y$	$\frac{dv_y}{dt} \equiv a_y$	$mv_y \equiv p_y$	$\frac{dp_y}{dt} \equiv f_y$
X^3	z	$\frac{dz}{dt} \equiv v_z$	$\frac{dv_z}{dt} \equiv a_z$	$mv_z \equiv p_z$	$\frac{dp_z}{dt} \equiv f_z$
$g_{\mu\nu}X^\mu X^\nu$	r^2	v^2	a^2	p^2	$(ma)^2$

high speed, making $dp/dt \leq m\alpha$.

B. Content modernization

Hestenes[11], Taylor[12], Moore and Schroeder[13] among others, have discussed and helped to address the challenge we face of modernizing content in spite of the fact that those who purchase textbooks like to teach it the way they learned it. In that context this article encourages its readers to help take responsibility for improvement of pedagogically-useful material in joint-editing spaces on campus and on the web, including Wikipedia's pages on proper-velocity and proper-acceleration.

II. THE TOOLS

The conceptual tools needed are basically time-derivatives, frame-invariance, and mass. These are tools that students of introductory physics may already be putting to use.

A. Newtonian recap

Newtonian physics is naturally introduced by starting with a Cartesian reference frame of co-moving yardsticks to measure map position coordinates $\vec{r} = \{x, y, z\}$ and co-moving synchronized clocks measuring map-time t with which to describe motion. One also needs a notion of distance which is independent one's choice of Cartesian axes. This comes from Pythagoras' assertion that the frame-invariant distance between two points in three-dimensional space at time t is given by:

$$(\delta r)^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2. \quad (1)$$

Given these tools, one takes derivatives with respect to map-time t so as to similarly define vector velocities and accelerations as shown in Table I.

The bottom row of quantities in this table are minimally frame-dependent scalars only to the extent that map-time t is frame-invariant. Newton had no way of knowing that this condition is not met for frames moving at sufficiently high speed with respect to one another.

Newton's dynamical laws then emerge by multiplying velocity by frame-invariant mass m to get momentum \vec{p} , and noting that changes in this quantity require interaction (namely an impulse-exchange) with the world around. This leads to:

$$\Sigma_i \vec{f}_i = \frac{d\vec{p}}{dt} = m \frac{d\vec{v}}{dt} = m\vec{a} \quad (2)$$

along with the very powerful scalar work-energy integral (low-speed version):

$$\frac{W}{m} = \vec{a} \cdot \Delta\vec{x} = \frac{1}{2} (v^2 - v_o^2) \quad (3)$$

from which the principle of least-action[14] arises, along with Lagrangian/Hamiltonian dynamics and the Feynman propagator[15].

As we see below, relativistic dynamics flows from this same sequence of steps simply by extending Pythagoras' theorem to include time-elapsd on the clocks of a moving observer.

B. The new variable

The simplest way to prepare for what's next is to ask students to specify *which clock* as well as *which yardsticks* when they identify the reference frame they'll use to describe motion. For the most part in Newtonian physics class this means defining time t as the time-elapsd on a set of synchronized clocks co-moving with those yardsticks.

By specifying the set of synchronized (or "map") clocks used, course participants will get in the habit of making problems well-posed in terms of what we now know. This also sets the stage early-on for introduction of **the one new variable needed for all of special relativity**, namely the frame-invariant time-elapsd (i.e. the proper-time τ) on the clocks of a moving object. This variable is given meaning through Minkowski's extension of Pythagoras' theorem.

C. The added equation

Minkowski's space-time extension of Pythagoras' theorem, namely the flat-space metric equation, might then be described in Cartesian coordinates as:

TABLE II: Cartesian 4-vector relationships \Rightarrow dynamics in (3+1)D spacetime.

component X^μ	4-position R R^μ	4-velocity V $U^\mu \equiv \frac{dR^\mu}{d\tau}$	4-acceleration A $A^\mu \equiv \frac{dU^\mu}{d\tau}$	momENergy cP $cP^\mu \equiv cmU^\mu$	net 4-force ΣF $\Sigma F^\mu \equiv mA^\mu$
X^0	ct	$c\frac{dt}{d\tau} \equiv c\gamma = \frac{c}{\sqrt{1-(\frac{v}{c})^2}}$	$c\frac{d\gamma}{d\tau} = \gamma^4 \frac{\vec{a} \cdot \vec{v}}{c}$	$\gamma mc^2 \equiv E$	$\frac{1}{c} \frac{dE}{d\tau} \equiv \gamma \frac{\vec{f} \cdot \vec{v}}{c}$
X^1	x	$\frac{dx}{d\tau} \equiv w_x = \gamma v_x$	$\frac{dw_x}{d\tau} = \gamma^2 a_x + \gamma^4 \frac{\vec{a} \cdot \vec{v}}{c} \frac{v_x}{c}$	$cmw_x \equiv cp_x$	$\frac{dp_x}{d\tau} \equiv \gamma f_x$
X^2	y	$\frac{dy}{d\tau} \equiv w_y = \gamma v_y$	$\frac{dw_y}{d\tau} = \gamma^2 a_y + \gamma^4 \frac{\vec{a} \cdot \vec{v}}{c} \frac{v_y}{c}$	$cmw_y \equiv cp_y$	$\frac{dp_y}{d\tau} \equiv \gamma f_y$
X^3	z	$\frac{dz}{d\tau} \equiv w_z = \gamma v_z$	$\frac{dw_z}{d\tau} = \gamma^2 a_z + \gamma^4 \frac{\vec{a} \cdot \vec{v}}{c} \frac{v_z}{c}$	$cmw_z \equiv cp_z$	$\frac{dp_z}{d\tau} \equiv \gamma f_z$
$g_{\mu\nu} X^\mu X^\nu$	$\rho^2 \equiv -(c\tau)^2$	$-c^2$	α^2	$-(mc^2)^2$	$(m\alpha)^2$

$$(c\delta t)^2 - (\delta r)^2 = \begin{cases} (c\delta\tau)^2 & \text{if } \delta r < c\delta t \text{ (timelike)} \\ 0 & \text{if } \delta r = c\delta t \text{ (null)} \\ -(\delta\rho)^2 & \text{if } \delta r > c\delta t \text{ (spacelike)} \end{cases} \quad (4)$$

Just as Pythagoras' theorem provides a frame-invariant measure of the distance between two points at a given map-time t , this extension of that equation yields a measure of the separation in space-time between two events which is independent of reference-frame as well. Here events are specified by a map-time coordinate-value (in distance units written as ct where c is the space-time constant commonly called lightspeed) as well as Cartesian position values for $\{x, y, z\}$.

In an introductory physics class, this equation might only be mentioned in passing as the reason one needs to specify "which clock" when asking how much time has elapsed. Once traveler-time has been defined in this way, it would let you later point to "anyspeed versions" e.g. of the constant acceleration and force equations discussed here even if there is no time to cover them in class.

In third-semester introductory and/or modern physics classes, this equation is a natural for exploring time-dilation. To wit: If a car with a clock on its roof is shown by video surveillance cameras to pass one bank when it and the bank clock both read 2:02, and to pass a second bank at 2:06 when the car clock only reads 2:04, how fast was the car going? An even more interesting if time-consuming prospect might be to start by giving students access to experimental data[16] on clock differences at high speed, modeling workshop style, to see what concepts *they* might use to quantify these effects.

D. Taking derivatives

As we did above when recapping Newtonian dynamics, it's instructive to tabulate the derivatives of an event's coordinates $\{ct, x, y, z\}$. In Table II we take first and second proper-time τ derivatives of the displacement between events, and then we multiply each by the traveling object's invariant mass m :

In addition to the now-familiar metric equation invariant for 4-displacement R, this table yields frame-invariants for 4-velocity U, 4-acceleration A, momENergy (the energy-momentum 4-vector) cP, and net-4-force (the

power-force 4-vector) ΣF . Key dynamical results that follow directly are that relativistic momentum $\vec{p} = m\vec{w} = m\gamma\vec{v}$, and relativistic energy $E^2 = (mc^2)^2 + (cp)^2$.

Another more general result that emerges is the fact that the **magnitude-squared of a 4-vector** has different properties than the magnitude-squared of a 3-vector. First of all, the magnitude-squared can be positive (space-like) or negative (time-like). Similarly, the absolute size of the magnitude also has different meaning.

For instance the absolute value of an object's 4-velocity is always c , whether it's moving or not! Speeding up changes its direction through space-time, but not the 4-velocity magnitude itself.

Similarly the magnitude of an object's four-vector acceleration **does** tell us about its felt-acceleration with respect to a local free-float frame. However it **does not** give us directly the net sum of forces (i.e. rates of momentum-change) seen from a differently-moving frame, since $d\gamma/d\tau$ and the rate of an accelerated object's energy-change is only zero from the vantage point of a co-moving frame where γ bottoms out at the value 1.

Note that rates of energy-change are frame-dependent (and zero from the co-moving perspective) even in Newtonian physics. Surveys I've taken at national AAPT meetings suggest that this is not intuitively-obvious to many physics teachers, who may be used to thinking that dynamical properties (like force) are frame-independent.

The following 4-vector facts should be easy to verify from Table II: The magnitude of an object's 4-velocity (displacement's proper-time derivative) is always *time-like* with magnitude c , while the magnitude of its 4-acceleration is always *space-like* with magnitude equal to proper-acceleration α . Similarly the magnitude of an object's momENergy is always time-like with magnitude mc^2 , while the magnitude of its net-4-force is always space-like with magnitude equal to the net-proper-force $m\alpha$. Note also that rates of momentum and energy change in a given frame are not frame-independent, and therefore that frame-variant force components defined directly in terms of momentum and energy change come in handy along with net-proper-force at high speeds.

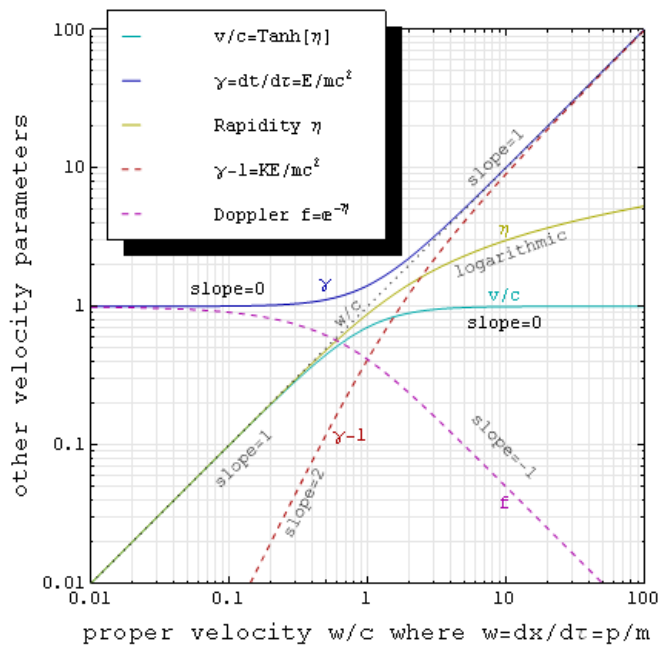


FIG. 1: Log-log plot of γ , $\frac{v}{c}$, and η vs. proper velocity $\frac{w}{c}$.

III. THE 1ST DERIVATIVE AND PROPER-VELOCITY

Proper-velocity, the distance traveled per unit time elapsed on the clocks of a traveling object, equals coordinate velocity at low speeds. Proper velocity at high speeds, moreover, retains many of the properties that coordinate-velocity loses.

For example proper-velocity equals momentum per unit mass at any speed, and therefore has no upper limit. At high speeds, as shown in Figure 1, it is proportional to an object's energy as well.

The full table of four-vector relationships above is not needed to introduce proper-velocity. For example simply taking the derivative of the metric equation itself with respect to proper time τ , to describe the timelike worldline of a traveler, yields:

$$\left(c \frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 = c^2 \quad (5)$$

and therefore

$$\gamma \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \sqrt{1 + \left(\frac{w}{c}\right)^2}, \quad (6)$$

where coordinate-velocity $\vec{v} \equiv d\vec{r}/dt$ and proper-velocity $\vec{w} \equiv d\vec{r}/d\tau = \gamma\vec{v}$.

Thus thanks to the metric-equation's assignment of a frame-invariant traveler or proper-time τ to the displacement between events in context of a single map-frame

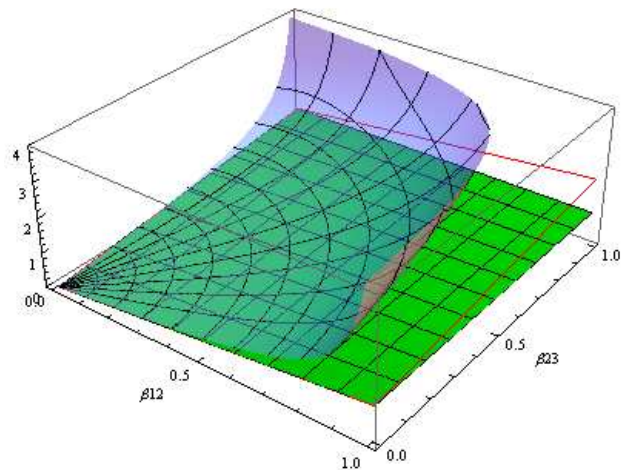


FIG. 2: Unidirectional velocity addition.

of co-moving yardsticks and synchronized clocks, proper-velocity becomes one of three related derivatives in special relativity (coordinate velocity \vec{v} , proper-velocity \vec{w} , and Lorentz factor γ) that describe an object's rate of travel. For unidirectional motion, in units of lightspeed c each of these is also simply related to a traveling object's hyperbolic velocity angle or rapidity η by

$$\eta = \sinh^{-1}\left[\frac{w}{c}\right] = \tanh^{-1}\left[\frac{v}{c}\right] = \pm \cosh^{-1}[\gamma] \quad (7)$$

A. Speed table

Table III illustrates how the proper-velocity of $w_o \equiv c$ or "one map-lightyear per traveler-year" is a natural benchmark for the transition from sub-relativistic to super-relativistic motion.

Note from Figure 1 that velocity angle η and proper-velocity w run from 0 to infinity and track coordinate-velocity when $w \ll c$. On the other hand when $w \gg c$, proper-velocity tracks Lorentz factor γ while velocity angle η is logarithmic and hence increases much more slowly.

B. Unidirectional velocity "addition"

For unidirectional motion, at low speeds the velocity v_{13} of object 1 from the point of view of oncoming object 3 might be described as the sum of the velocity v_{12} of object 1 with respect to lab frame 2 plus the velocity v_{23} of the lab frame 2 with respect to object 3, i.e. as $v_{13} = v_{12} + v_{23}$. This Galilean addition rule is noted in Fig. 2 by the red outline.

At coordinate-speeds approaching c , coordinate-velocity deviates from this simple addition rule in that

TABLE III: Benchmark values around the relativistic slope-change in KE vs. momentum.

Parameter Condition	Coordinate-velocity v [ly/map-year]	Velocity-angle η [hyperbolic-radians]	Proper-velocity w [ly/trav-year]	Lorentz factor γ [map-yr/trav-yr]
traveler stopped	0	0	0	1
momentum $\frac{1}{2}mc$	$\frac{1}{\sqrt{5}} \simeq 0.447$	$\ln[\frac{1+\sqrt{5}}{2}] \simeq 0.481$	$\frac{1}{2}$	$\frac{\sqrt{5}}{2} \simeq 1.118$
rapidity $\frac{1}{2}$ v -radian	$\frac{e-1}{e+1} \simeq 0.462$	$\frac{1}{2}$	$\frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2} \simeq 0.521$	$\frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}{2} \simeq 1.128$
coord-vel $v = \frac{1}{2}c$	$\frac{1}{2}$	$\frac{1}{2} \ln[3] \simeq 0.549$	$\frac{1}{\sqrt{3}} \simeq 0.577$	$\frac{2}{\sqrt{3}} \simeq 1.155$
momentum mc	$\frac{1}{\sqrt{2}} \simeq 0.707$	$\ln[1 + \sqrt{2}] \simeq 0.881$	1	$\sqrt{2} \simeq 1.414$
rapidity 1 v -radian	$\frac{e^2-1}{e^2+1} \simeq 0.761$	1	$\frac{e-1/e}{2} \simeq 1.175$	$\frac{e+1/e}{2} \simeq 1.543$
kinetic energy mc^2	$\frac{\sqrt{3}}{2} \simeq 0.866$	$\ln[\sqrt{3} + 2] \simeq 1.317$	$\sqrt{3} \simeq 1.732$	2
momentum $2mc$	$\frac{2}{\sqrt{5}} \simeq 0.894$	$\ln[2 + \sqrt{5}] \simeq 1.444$	2	$\sqrt{5} \simeq 2.236$
rapidity 2 v -radian	$\frac{e^4-1}{e^4+1} \simeq 0.964$	2	$\frac{e^2-1/e^2}{2} \simeq 3.627$	$\frac{e^2+1/e^2}{2} \simeq 3.762$
coord-vel $v = c$	1	∞	∞	∞

rapidities (hyperbolic velocity angle boosts) add instead of velocities, i.e. $\eta_{13} = \eta_{12} + \eta_{23}$. As a result coordinate-velocity follows the green checkerboard in the figure, while proper-velocity (that transparent sheet) goes through the roof.

For highly relativistic objects (i.e. with momentum per unit mass much larger than lightspeed) the result of the coordinate-velocity expression (green mesh in the figure at right) familiar from most textbooks is rather uninteresting since the coordinate-velocities all peak out at c , i.e. $(c+c)/(1+1) \Rightarrow c$.

For relative proper-velocity, the result is:

$$w_{13} = \gamma_{13}v_{13} = \gamma_{12}\gamma_{23}(v_{12} + v_{23}). \quad (8)$$

This expression shows how the momentum per unit mass as well as the map-distance traveled per unit traveler time of object 1, as seen in the frame of oncoming particle 3, goes as *the sum of the coordinate-velocities times the product of the gamma (energy) factors*. This is denoted by the sky-ward climbing radial mesh in the figure.

The proper-velocity equation is especially important in high energy physics, because colliders enable one to explore proper-speed and energy ranges much higher than accessible with fixed-target collisions. For instance each of two electrons (traveling with frames 1 and 3) in a head-on collision traveling in the lab frame (2) at $\gamma_{12}mc^2 = 45[\text{GeV}]$ or $w_{12} = w_{23} = \gamma v \simeq 88,000[\text{lightseconds per traveler second}]$ would see the other coming toward them at $v_{13} \simeq c$ and $w_{13} = 88,000^2(1+1) \simeq 1.55 \times 10^{10}[\text{lightseconds per traveler second}]$ or $\gamma_{13}mc^2 \simeq 7.9[\text{PeV}]$. From the target's point of view, that's quite an increase in both energy and momentum per unit mass!

IV. THE 2ND DERIVATIVE AND PROPER-ACCELERATION

Proper-acceleration is the physical acceleration experienced by an object relative to a locally co-moving

free-float frame. This may be quite different than the coordinate-acceleration seen in a separate reference frame (inertial or not) of co-moving yardsticks and synchronized clocks.

The proper-acceleration 3-vector, combined with a null time-component, yields the object's four-acceleration (as measured by the object itself) which makes proper-acceleration's magnitude Lorentz-invariant. Thus proper-acceleration comes in handy: (i) with accelerated coordinate systems, (ii) at relativistic speeds, and (iii) in curved spacetime.

Proper-acceleration reduces to coordinate-acceleration in an inertial coordinate system in flat spacetime (i.e. in the absence of gravity), provided the magnitude of the object's proper-velocity (momentum per unit mass) is much less than the speed of light c . Here we focus on situations where proper-acceleration and coordinate-acceleration are not always the same.

A. In accelerated frames

The intro-physics caution about using Newton's Laws only in un-accelerated or inertial frames is a strategy for avoiding confusion by non-proper (i.e. "geometric") coordinate-accelerations like those experienced e.g by passengers in vehicles accelerated from a stop-sign or going around a curve. As indicated below, the equivalence principle of general relativity in fact shows that Newton's laws work locally in any coordinate system provided one invokes a suitable set of "geometric" or affine-connection accelerations/forces.

This result validates the intro-physics practice of treating shell-frames on planet earth as free-float-frames so that gravity becomes an external force. It also validates local reference to other "fictitious" forces like Coriolis and centrifugal.

The rule of thumb in each case is that non-proper or "geometric" accelerations act on every ounce of an object's mass, and vanish if one describes the motion from the perspective of a free-float-frame moving with

our accelerated object. Thus the concept of proper-acceleration, as the physical-acceleration experienced by an object with reference to a locally-comoving free-float-frame, is already an important introductory physics tool.

B. At high speeds

The time derivatives of proper-velocity in Table II show that momentum change (hence force) has an energy-change component in the velocity direction, proportional to $\vec{v} \cdot \vec{a}$, as well as the expected acceleration component in the direction of $\vec{a} \parallel \vec{a}$. Hence acceleration and force are no longer in the same direction.

However the last column in that table suggests a fairly simple dependence of proper-acceleration's *magnitude* α on force \vec{f} and velocity $\vec{v} \parallel \vec{w}$ vectors. In terms of components of velocity \vec{v} *transverse* (t) and *longitudinal* (ℓ) to the force vector \vec{f} , direct substitution gives:

$$\frac{m\alpha}{f} = \sqrt{\frac{1 - \left(\frac{v_t}{c}\right)^2}{1 - \left(\frac{v}{c}\right)^2}} = \sqrt{1 + \left(\frac{w_t}{c}\right)^2} \equiv \gamma_t \geq 1. \quad (9)$$

As you can see the equality obtains in the high-speed unidirectional, as well as the low-speed, limits.

Since the angle between force and acceleration varies, a more detailed analysis is needed in order to compare their *directions*. One might begin by writing the Lorentz factor in terms of velocities parallel and perpendicular to the acceleration direction, i.e.

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v_{\parallel}}{c}\right)^2 - \left(\frac{v_{\perp}}{c}\right)^2}} \Rightarrow c \frac{d\gamma}{dt} = \gamma^3 \left(\frac{v_{\parallel}}{c}\right) \frac{dv_{\parallel}}{dt}. \quad (10)$$

From this it's relatively easy to calculate the proper-velocity derivatives:

$$\frac{dw_{\parallel}}{dt} = \gamma \frac{dv_{\parallel}}{dt} + v_{\parallel} \frac{d\gamma}{dt} = \gamma^3 \frac{dv_{\parallel}}{dt} \left(1 - \frac{v_{\perp}^2}{c^2}\right) \quad (11)$$

and

$$\frac{dw_{\perp}}{dt} = v_{\perp} \frac{d\gamma}{dt} = \gamma^3 \frac{dv_{\parallel}}{dt} \left(\frac{v_{\perp}}{c} \frac{v_{\parallel}}{c}\right). \quad (12)$$

Converting these to derivatives with respect to proper-time on multiplication by $\gamma \equiv dt/d\tau$, one can then form the invariant 4-vector sum α^2 to get:

$$\alpha = \sqrt{\left(\frac{dw_{\parallel}}{d\tau}\right)^2 + \left(\frac{dw_{\perp}}{d\tau}\right)^2 - \left(c \frac{d\gamma}{d\tau}\right)^2} = \frac{\gamma^3}{\gamma_{\perp}} \frac{dv_{\parallel}}{dt}. \quad (13)$$

Thus if our observer is moving at fixed velocity v_{\perp} "perpendicular" to the path of a uniformly-accelerated traveler, derivatives with respect to map-time t can be written in terms of the frame-invariant proper-acceleration α as:

$$\frac{dw_{\parallel}}{dt} = \frac{\alpha}{\gamma_{\perp}} \text{ and } \frac{dw_{\perp}}{dt} = c \frac{d\gamma}{dt} \left(\frac{v_{\perp}}{c}\right), \quad (14)$$

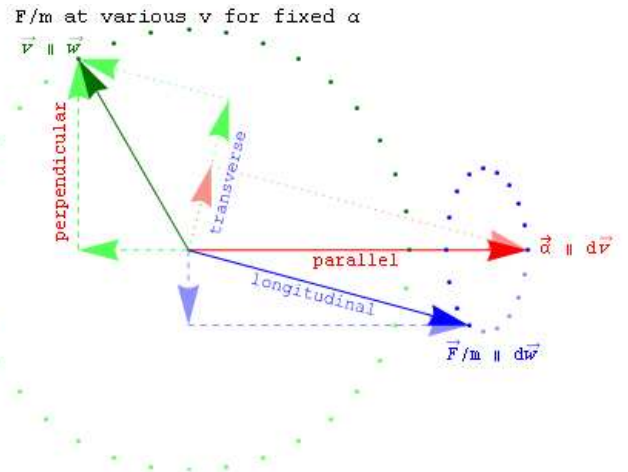


FIG. 3: Proper-acceleration components for $v = 0.65c$.

where

$$c \frac{d\gamma}{dt} = \alpha \gamma_{\perp} \left(\frac{v_{\parallel}}{c}\right). \quad (15)$$

Here $w_{\parallel} \equiv dx_{\parallel}/dt = \gamma v_{\parallel}$ is the component of proper-velocity parallel to the proper-acceleration direction, while $\gamma_{\perp} \equiv 1/\sqrt{1 - (v_{\perp}/c)^2}$ and $v_{\perp} \equiv dx_{\perp}/dt = w_{\perp}/\gamma$ are respectively the Lorentz-factor and the (unchanging) component of coordinate-velocity perpendicular to the same.

From this one can write the integrals of constant proper-acceleration in (3+1)D spacetime as:

$$\alpha = \gamma_{\perp} \frac{\Delta w_{\parallel}}{\Delta t} = c \frac{\Delta \eta_{\parallel}}{\Delta \tau} = \frac{c^2}{\gamma_{\perp}} \frac{\Delta \gamma}{\Delta x_{\parallel}} \quad (16)$$

where $\eta_{\parallel} \equiv \sinh^{-1}[w_{\parallel}/c]$. These reduce to the unidirectional proper-acceleration equations

$$\alpha = \frac{\Delta w}{\Delta t} = c \frac{\Delta \eta}{\Delta \tau} = c^2 \frac{\Delta \gamma}{\Delta x} \quad (17)$$

when $v_{\perp} = 0 \Rightarrow \gamma_{\perp} = 1$, and to the Newtonian equations for coordinate-acceleration with respect to an inertial frame in three-dimensions when $v \ll c$.

C. Why is $f \leq m\alpha$?

From the foregoing analysis one can also express Newton's 2nd Law in (3+1)D spacetime for a single (frame-variant) force or net-force by writing[17]:

$$\vec{f} \equiv \frac{d\vec{p}}{dt} = \frac{m\alpha}{\gamma_t} \hat{i}_{\ell} \quad (18)$$

where the unit vector *longitudinal* to the frame-variant force direction is:

$$\hat{i}_{\ell} \equiv \left(\frac{\gamma_t}{\gamma_{\perp}}\right) \hat{i}_{\parallel} + \left(\gamma_t \gamma_{\perp} \frac{v_{\perp}}{c} \frac{v_{\parallel}}{c}\right) \hat{i}_{\perp}. \quad (19)$$

TABLE IV: Various quantities ranked according to their frame-(in)dependence.

quantity	frame invariant	4-vector component	effective for the indicated frame only (eifo)
time	proper (clock-frame) time $\Delta\tau$	map-time $\Delta t = \gamma\Delta\tau$	
length	proper (rest-frame) length $\Delta\rho$	map-distance Δr	contracted length $\Delta\rho/\gamma$
speed of map-time and total energy		Lorentz factor $\gamma \equiv dt/d\tau$ $E = \gamma mc^2$	
3-vector velocity and momentum		proper-velocity $\vec{w} \equiv d\vec{r}/d\tau$ momentum $\vec{p} = m\vec{w}$	coordinate-velocity $\vec{v} \equiv d\vec{r}/dt = \vec{w}/\gamma$
3-vector acceleration and proper-force	proper-acceleration α proper-force $\Sigma\vec{F} \equiv m\vec{\alpha}$		coordinate-acceleration $\vec{a} \equiv d\vec{v}/dt = (\gamma_\perp/\gamma^3)\vec{\alpha}$
rate of momentum change		$d\vec{p}/d\tau$	net-force $\Sigma\vec{f} \equiv d\vec{p}/dt = (\Sigma\mathcal{F}/\gamma_t)\hat{i}_\ell$
rate of energy change		$dE/d\tau$	power $dE/dt = \Sigma\vec{f} \cdot \vec{v}$

Here the gamma-value transverse to the frame-variant force direction can be written:

$$\gamma_t \equiv \sqrt{1 + \left(\frac{w_t}{c}\right)^2} = \frac{1}{\sqrt{\left(\frac{1}{\gamma_\perp}\right)^2 + \left(\frac{\gamma_\perp v_\perp}{c} \frac{v_\parallel}{c}\right)^2}}. \quad (20)$$

Conversely for any observed rate of momentum-change (or combination of frame-variant forces) one can associate 3-vectors with the scalar frame-invariants proper-acceleration α and proper-force $\mathcal{F} \equiv m\alpha$. These obey:

$$\vec{\mathcal{F}} \equiv m\vec{\alpha} = f\gamma_t\hat{i}_\parallel \quad (21)$$

where the unit vector in the proper-acceleration direction is:

$$\hat{i}_\parallel \equiv \left(\frac{\gamma_t}{\gamma_\perp}\right)\hat{i}_\ell + \left(\frac{1}{\gamma_\perp\gamma_t} \frac{w_t}{c} \frac{w_\ell}{c}\right)\hat{i}_t \quad (22)$$

and subscript t denotes quantities *transverse* to the frame-variant force.

The magnitude of the proper force is greater than or equal to the force seen in an observer-frame, i.e. $\mathcal{F} \equiv m\alpha \geq f \equiv dp/dt$, with equality *at any speed* as long as the accelerated object's velocity transverse to the force (i.e. w_t) is zero.

As we show below the proper-force $\mathcal{F}_z = m\alpha$ on a charge moving sideways at speed v_x with respect to some protons at rest **is least** when $v_x = 0$, since the reference-frame force $f \equiv dp/dt$ is independent of v_x . If on the other hand the proper force were independent of v_x , e.g. in the case of a constant proper-acceleration rocket-ship, then **the reference-frame force peaks** when $v_x = 0$.

In both cases one can think of this as the force dilution (or “eifo-weakening” ala Shurcliff) that takes place as *changes in* proper-velocity (hence momentum) are lost to the reference-frame's time-domain because of sideways motion. Since this stems from the frame-invariance of proper-acceleration, it's really just another metric-based kinematic-effect like time-dilation (eifo-slowness) and length-contraction (eifo-shortening).

1 constant acceleration example

As a function of traveler or proper time τ , for constant proper-acceleration in the x_\parallel direction from rest at the origin we might write the traveler's map time and position coordinates, or 4-vector displacement R^μ , as:

$$R^0 = ct = \frac{c^2}{\alpha} \left(\gamma_\perp \sinh \left[\frac{\alpha\tau}{c} \right] \right). \quad (23)$$

$$R^1 = x_\parallel = \frac{c^2}{\alpha} \left(\cosh \left[\frac{\alpha\tau}{c} \right] - 1 \right) \geq 0. \quad (24)$$

$$R^2 = x_\perp = \frac{c^2}{\alpha} \left(\frac{\gamma_\perp v_\perp}{c} \sinh \left[\frac{\alpha\tau}{c} \right] \right). \quad (25)$$

Taking derivatives and calculating the magnitude of $d\vec{p}/dt \equiv md\vec{w}/dt$, one gets:

$$f \equiv \frac{dp}{dt} = m\alpha \sqrt{1 - \left(\frac{v_\perp}{c}\right)^2} \left(1 - \tanh \left[\frac{\alpha\tau}{c} \right]^2 \right). \quad (26)$$

As predicted, $f \leq m\alpha$.

2 electromagnetic example

A fixed charge Q acts on a moving charge q via the radial frame-variant force $f = kqQ/r^2$. Here we choose the rest frame of the source charge so that the field is stationary. The “frame-invariant” but velocity-dependent proper-force $\mathcal{F} \equiv m\alpha$ on *any moving charge* from above is then:

$$\vec{\mathcal{F}} = k \frac{qQ}{r^2} \sqrt{1 + \left(\frac{w_\phi}{c}\right)^2} \hat{i}_\parallel, \quad (27)$$

where

$$\hat{i}_{\parallel} \equiv \frac{\left(1 + \left(\frac{w_{\phi}}{c}\right)^2\right) \hat{i}_r + \left(\frac{w_{\phi}}{c} \frac{w_r}{c}\right) \hat{i}_{\phi}}{\sqrt{\left(1 + \left(\frac{w_{\phi}}{c}\right)^2\right)^2 + \left(\frac{w_{\phi}}{c} \frac{w_r}{c}\right)^2}}. \quad (28)$$

This velocity dependence of the proper force in (3+1)D space-time gives rise to relativistic effects for almost all forces, and to magnetism from Coulomb's Law.

To illustrate this, consider the simpler special case when the force is perpendicular to the line of a test particle's velocity $v_{\perp} = v_{\phi}$. Then the proper-force \mathcal{F} is simply related to the frame-variant force f (defined as usual in terms of momentum and energy transfer) by:

$$\vec{\mathcal{F}} = k \frac{qQ}{r^2} \sqrt{1 + \left(\frac{w_{\phi}}{c}\right)^2} \hat{i}_{\parallel} = f \gamma_{\perp} \hat{r}. \quad (29)$$

This lets one use symmetry and the frame-invariance of proper force in a single frame, instead of length contraction and two-frames ala Purcell[18], to illustrate the connection between relativity and magnetism. For example consider a horizontal and electrically-neutral wire element ds with $-\lambda$ Coulomb of free electrons per unit length moving at speed v_x to the right. Their excess charge is cancelled out by the same number of bound protons per unit length, standing still. The current in the wire (defined as moving in the -x direction) is $I = \lambda v_x$.

For a **stationary positive charge q** a distance r above the wire, $\gamma_{\perp} = \gamma_x$ is 1 and the net upward (repulsive) force is:

$$f_z = \mathcal{F}_z = \underbrace{q(E_o)}_{\text{proton } \mathcal{F}} + \underbrace{q(-E_o)}_{\text{electron } \mathcal{F}} = 0 \quad (30)$$

where the base electric field magnitude $E_o \equiv k\lambda ds/r^2$.

However if **the same positive charge q is now moving to the right** with those electrons, the proper-force exerted on our test charge by the stationary positive charges remains in the direction of their separation but *increases* in magnitude by that factor of γ_x . What the moving electrons do to the test particle as it matches speed with them would be less clear, except that by symmetry the frame-invariant proper-force in that case must *decrease* by the same factor of γ_x since the test particle now moves with them rather than the protons.

Hence the net upward *Coulomb's Law force* from the metric equation becomes:

$$f_z = \frac{1}{\gamma_x} \mathcal{F}_z = \frac{1}{\gamma_x} \left(\underbrace{q(E_o)\gamma_x}_{\text{proton } \mathcal{F}} + \underbrace{q(-E_o)\frac{1}{\gamma_x}}_{\text{electron } \mathcal{F}} \right) \quad (31)$$

so that

$$f_z = qE_o \left(1 - \frac{1}{\gamma_x^2}\right) = qE_o \left(\frac{v_x}{c}\right)^2. \quad (32)$$

If we define magnetic field using Biot-Savart as

$$B_y \equiv \frac{k}{c^2} \frac{Ids}{r^2} = \frac{k}{c^2} \frac{\lambda v_x ds}{r^2} = \frac{v_x E_o}{c^2}, \quad (33)$$

then this analysis yields the Lorentz force equation:

$$f_z = qv_x B_y. \quad (34)$$

D. Curvature and equivalence

In the language of general relativity, the components of an object's acceleration four-vector A (whose magnitude is proper acceleration) are related to elements of the four-velocity via a covariant derivative \mathcal{D} with respect to proper time τ :

$$A^{\lambda} = \frac{\mathcal{D}U^{\lambda}}{d\tau} = \frac{dU^{\lambda}}{d\tau} + \Gamma^{\lambda}_{\mu\nu} U^{\mu} U^{\nu}. \quad (35)$$

Here U is the object's four-velocity, and Γ represents the coordinate system's 64 connection coefficients or Christoffel symbols. Note that the Greek subscripts take on four possible values, namely 0 for the time-axis and 1-3 for spatial coordinate axes, and that repeated indices are used to indicate summation over all values of that index. Trajectories with zero proper acceleration are referred to as geodesics or free-float trajectories.

Thus coordinate acceleration goes to zero whenever proper-acceleration is exactly canceled by the connection (or "geometric acceleration") term on the far right[8]. *Caution:* This term may be a sum of as many as sixteen separate velocity and position dependent terms, since the repeated indices μ and ν are by convention summed over all pairs of their four allowed values.

The above equation also offers some perspective on forces and the equivalence principle. Consider "local" book-keeper coordinates[3] for the metric (e.g. a local Lorentz tetrad[19] like that on which global positioning systems provide information) to describe time in seconds, and space in distance units along perpendicular axes. If we multiply the above equation by the traveling object's rest mass m , and divide by Lorentz factor $\gamma \equiv dt/d\tau$, the spacelike components express the rate of momentum change for that object from the perspective of the coordinates used to describe the metric.

This in turn can be broken down into parts due to proper and geometric components of acceleration and force. If we further multiply the time-like component by lightspeed c , and define coordinate velocity as $\vec{v} \equiv d\vec{x}/dt$, we get an expression for rate of energy change as well:

$$\frac{dE}{dt} = \vec{v} \cdot \frac{d\vec{p}}{dt} \text{ (timelike)} \quad (36)$$

and

$$\frac{d\vec{p}}{dt} = \Sigma \vec{f}_o + \Sigma \vec{f}_g = m(\vec{a}_o + \vec{a}_g) \text{ (spacelike)}. \quad (37)$$

Here a_o is an *acceleration term* (not necessarily a physical acceleration) due to proper forces and a_g is, by default, a geometric acceleration term that we see applied to the object because of our coordinate system choice. At low speeds these accelerations combine to generate a coordinate acceleration like $\vec{a} = d^2\vec{x}/dt^2$, while for uni-directional motion at any speed a_o 's magnitude is that of proper acceleration α as above. In general expressing these accelerations and forces can be complicated.

Nonetheless if we use this breakdown to describe the connection coefficient (Γ) term above in terms of geometric forces, then the motion of objects from the point of view of any coordinate system (at least at low speeds) can be seen as locally Newtonian. This is already common practice e.g. with gravity and centrifugal force.

V. DISCUSSION AND CONCLUSIONS

Einstein's writings on special relativity, which served to guide early texts, were inspired by the frame-invariance of lightspeed. Thus extended frames in relative motion were front and center. However just as Einstein's impression of Minkowski's insight as *überflüssige gelehrsamkeit*[20] (superfluous erudition) was eclipsed by its adoption as key to a geometric understanding of gravitation, so recognition of privileged frames (e.g. the proper-time frame of a moving clock, and the rest-frame of a moving yardstick) is becoming more and more common in modern texts.

The late William A. Shurcliff's passion for clear and minimally-variant descriptions, like that of the metric equation with its "moving clock time $d\tau$ ", can be used to take this trend further as outlined in Table IV. But textbooks alone (as time has shown) cannot lead the way to simpler and clearer presentations. Shared materials in joint-editing space, however, can give energetic teachers grass-roots input into the evolution of content. Responsibility for participation in the process, of course, lies with the reader.

In summary, the behavior of spacetime at high speeds does not have to be introduced by expanding the dreaded "relative motion section" of an introductory physics course to include Lorentz transforms. An alternative is to introduce the traveler-time variable τ via the metric equation. Given this, all of special relativity including Lorentz transforms[6] follows via a single-frame narrative patterned after that used already to introduce Newtonian dynamics.

The frame-invariants that emerge for describing motion, namely lightspeed c and proper-acceleration α , are scalars even though proper-acceleration in the free-float frame co-moving with an accelerated traveler can be broken nicely into three space-like components as well. Since they are true frame-invariants, these scalars are useful in curved-spacetime too. Similarly, as William Shurcliff put it in a private communication, proper-velocity is speed "defined in a manner which requires no synchrony". No

wonder it equals momentum per unit mass, and represents three of four components of a 4-vector whose time-like component is $c\gamma$.

The metric equation is sometimes mentioned at the end of introductory sections on relativity. It's argued here that, with help from proper-time, proper-velocity, and proper-acceleration, the metric equation might instead be used from the start.

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