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Compactly supported tight and sibling frames with maximum vanishing moments ☆

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Abstract

The notion of vanishing-moment recovery (VMR) functions is introduced in this paper for the construction of compactly supported tight frames with two generators having the maximum order of vanishing moments as determined by the given refinable function, such as the mth order cardinal B-spline N_m. Tight frames are also extended to "sibling frames" to allow additional properties, such as symmetry (or antisymmetry), minimum support, "shift-invariance," and inter-orthogonality. For N_m, it turns out that symmetry can be achieved for even m and antisymmetry for odd m, that minimum support and shift-invariance can be attained by considering the frame generators with two-scale symbols $2^{-m}(1-z)^m$ and $2^{-m}z(1-z)^m$, and that inter-orthogonality is always achievable, but sometimes at the sacrifice of symmetry. The results in this paper are valid for all compactly supported refinable functions that are reasonably smooth, such as piecewise $\mathrm{Lip}\,\alpha$ for some $\alpha > 0$, as long as the corresponding two-scale Laurent polynomial symbols vanish at $z=-1$. Furthermore, the methods developed here can be extended to the more general setting, such as arbitrary integer scaling factors, multi-wavelets, and certainly biframes (i.e., allowing the dual frames to be associated with a different refinable function).

Keywords: Sibling frame; Tight frame; Unitary extension; Vanishing moment recovery; Inter-orthogonality; Matrix factorization

1. Introduction

It is well known that symmetric or antisymmetric compactly supported real-valued orthonormal wavelets with dilation factor equal to 2 are integer translates of $\pm H$, where H denotes the Haar function [12]. In addition, again with the exception of these Haar functions $\pm H(\cdot - k)$, compactly supported orthonormal wavelets do not have explicit analytic formulation. However, in applications where certain function classes are needed to guarantee accuracy to be within certain range, such as 10^{-8} to 10^{-12} in representation of objects, or more importantly, to be compliant with certain industry standards, it is highly desirable to construct wavelets within the class of analytically representable functions.

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For instance, in the CAD/CAM (computer-aided design and manufacturing) industry, (polynomial) splines, and more generally NURBS, are used to represent curves and surfaces [26]. Therefore, when the wavelet approach is used to add features for such applications as editing, rendering, and oscillation measurement/correction to the spline tool-box of the CAD/CAM/CAE industry standards, particularly IGES and STEPS [26], it is more suitable to apply those wavelets that can be expressed as finite linear combinations of translates of the B-splines in the same parametric curve/surface representation space [34]. Semi-orthogonal spline wavelets [4,5] and biorthogonal spline wavelets [9,10,12] are the most natural candidates. However, both of these wavelets have undesirable duals. While the duals of semi-orthogonal spline wavelets have full support in the parametric domain [5], those of the biorthogonal spline wavelets are not in the same spline spaces.

Another option is to allow more than one wavelet generators. For example, compactly supported tight frames of mth order cardinal splines with m generators were introduced in [31]. In [6], it was shown that independent of the order m, two generators always suffice. The proof in [6] is constructive, and it is clear from the construction that the two filter lengths, or equivalently the degrees of the two-scale Laurent polynomial symbols associated with the mth order cardinal B-splines, are at most m. It was also shown in [6] that, again independent of m, at most three generators are sufficient to achieve symmetry/antisymmetry. With practical applications in mind, we strove to construct the minimum number of frame generators to meet such important requirements as compact support (or finite filter length), symmetry/antisymmetry (for linear-phase filtering), etc. Although it may be argued that more frame generators are perhaps desirable for yielding higher redundancy, yet in practical applications, when a (hardware/software) system is already built, it is no longer possible to reduce redundancy, when less redundancy is needed. By using a minimum number of (compactly supported) tight frame generators to design the system, arbitrarily higher redundancy can be easily achieved by adjusting the oversampling rate according to the specification of the second oversampling theorem of Chui and Shi [7], without the need of building a new (hardware/software) system. Recall that the second oversampling theorem guarantees preservation of tight frames.

However, regardless of the number of wavelet frame generators to be used, the "matrix extension" approach in [6,31] limits the order of vanishing moments to one, for at least one of the tight frame generators associated with the mth order cardinal B-spline, for $m \geqslant 2$. For applications that benefit from effective extraction of details, the order of vanishing moments is a key feature for the success of (analyzing) wavelets. In this paper, we introduce the notion of vanishing-moment recovery (VMR) functions for the construction of compactly supported tight wavelet frames to achieve the maximum order of vanishing moments as allowed by the order of (local) polynomial reproduction of the associated compactly supported refinable function. We again show that two frame generators always suffice. For example, with a VMR function, two compactly supported tight (spline-wavelet) frame generators associated with the mth order cardinal B-spline do indeed have the maximum mth order of vanishing moments. The work in this paper was motivated by the interesting paper [29] of Ron and Shen, where a complete characterization of tight frame (generators) is derived in terms of the so-called "fundamental function of multiresolution," again associated with some refinable function (see Theorem 6.5 in [29]). In fact, after the two-scale symbols of the tight frame generators have been constructed by using a VMR function, the VMR function indeed agrees with the fundamental function of Ron and Shen, which is defined in [29] in terms of the two-scale symbol of the refinable function as well as the two-scale symbols of the tight frame generators (that are to be constructed). The important distinction is that VMR functions are introduced in the present paper to construct the two-scale symbols of the frame generators.

When two compactly supported tight frame generators with the maximum number of vanishing moments (as allowed by the associated compactly supported refinable function) are constructed, there is no guarantee of symmetry (or antisymmetry). Another main objective of this paper is to introduce the notion of sibling frames. While tight frames may be considered as a natural generalization of orthonormal wavelets, the notion of sibling frames is introduced as a natural generalization of semiorthogonal wavelets in order to allow construction of compactly supported dual spline-wavelet frames. The additional flexibility provided by sibling non-tight frames is indeed sufficient to guarantee compact support, maximum order of vanishing moments, and symmetry (or antisymmetry), provided that the associated refinable function is compactly supported and symmetric. For certain applications, the sacrifice of tightness is worthwhile since on one hand, sibling frames are (finite) linear combinations of translates of the same refinable function, such as the same mth order cardinal B-splines, while on the other hand, their support can be made significantly smaller. Another important feature of sibling frames is that the two frame generators could be designed to be simply a shift of each other by $1/2$. This is significant in that the shift-variant defect of the standard wavelet decomposition procedure can be removed, even with downsampling. Recall that in a different context, Kingsbury [20,21] considered a dual tree of wavelet filters, where all the sampling rates of the fully decimated wavelet transform are doubled by eliminating the downsampling operation in the first decomposition step and where the filters of the subsequent decomposition steps are chosen with alternating parity, in order to achieve almost shift-invariant effect, with noticeable improvement in image denoising and texture analysis.

Another property that sibling frames can achieve is that the two frame generators can be designed to allow minimum correlation at the same (scale) level, in the sense that the two subspaces obtained by their integer shifts are orthogonal to each other. We call this property "inter-orthogonality." In applications to signal processing, a signal f is partitioned into "frequency bands" as identified by the different scale levels. The wavelet coefficients $d_{j,k,\ell}$ for each level, say level $j = j_0$, are the continuous (or integral) wavelet transforms of the signal f at the time-scale location $(k/2^{j_0}, 2^{-j_0})$, where the third subscript ℓ for $d_{j,k,\ell}$ specifies that the wavelet ψ_ℓ is used as the analysis wavelet. Hence, if ψ_1 and ψ_2 are inter-orthogonal frame generators, then the time-scale information $\{d_{j_0,k,1}\}$ and $\{d_{j_0,m,2}\}$, $k, m \in \mathbb{Z}$, of f separates the signal content $g_{j_0,1} + g_{j_0,2}$ of f on the level $j = j_0$ most efficiently, where $g_{j_0,\ell}$ is generated by $\psi_{j_0,k,\ell}$, $k \in \mathbb{Z}$, for $\ell = 1, 2$.

We remark that while the results in this paper are valid for biframes (i.e., by using two different refinable functions), we restrict our discussion to sibling and particularly tight frames, since we are particularly interested in compactly supported wavelets and dual wavelets of cardinal splines. In addition, our point of view is that if one allows two multiresolution analyses (or two refinable functions), one already has the well-known compactly supported symmetric/antisymmetric biorthogonal wavelets of Cohen et al. [9]; and again, oversampling can be applied to generate as much redundancy as desired [7]. On the other hand, although we use cardinal B-splines as a prototype quite frequently in our discussion, our results are more general. In fact, what is needed is only a compactly supported refinable function with unit integral and a very mild smoothness assumption, such as piecewise Lip α for some $\alpha > 0$, and such that its two-scale Laurent polynomial symbol has a factor $(1+z)^m$, $m \geqslant 1$. In particular, as in [29,30], the Riesz (or stability) condition is not required.

The following describes some of the main results obtained in this paper. Theorem 1 is devoted to the analysis of the VMR functions, with certain explicit formulations. In Theorem 2, particularly for mth order cardinal splines, existence of compactly supported sibling frames with two generators having mth order of vanishing moments and being symmetric or antisymmetric, depending on even or odd m, is established. In addition, the choice of frame generators with two-scale symbols

$$\left(\frac{1-z}{2}\right)^m \quad \text{and} \quad z\left(\frac{1-z}{2}\right)^m$$

is allowed. Note that this choice achieves minimum support and the "shift-invariance" property as mentioned above. The existence of sibling frames with two generators whose integer-translates constitute inter-orthogonal subspaces is established in Theorem 3. In addition, auxiliary and related results concerning sibling frames with one generator (Theorems 8 and 9), the matrix-valued Riesz Lemma (Theorem 4), and application of this lemma to establishing tight frames associated with stable refinable functions (Theorems 5 and 7) are also presented in this paper.

Related work concerning wavelet frames with higher vanishing moments has been carried out independently in parallel to our development by Daubechies et al. [14]. This article gives many interesting results on tight frames with several generators that are derived from a refinable function. They also give a refined proof of telescoping of the frame decomposition in [12], define a new notion of approximation order of tight frames, and describe how the fundamental function (i.e., the VMR function in our paper) affects the decomposition and reconstruction algorithms of tight frames. Moreover, they prove the existence of tight frames with two generators which are finite linear combinations of cardinal B-splines of arbitrary order. The fundamental function (or VMR function in our article) is used to achieve higher order of vanishing moments of all generators of the tight frame in [14]. A question raised in [14], that whether or not tight frames with several generators exist for any MRA, is answered affirmatively in our paper (see Theorems 5 and 7 for the univariate case with dilation factor 2), and we show that two generators are sufficient.

Two positivity conditions for the VMR functions S (or fundamental functions in [14]) are introduced, one in our present paper, and the other in the independent work [14]. The positivity condition in our paper is a linear formulation in $1/S$ which describes a necessary and sufficient condition for S to be a VMR function for all two-scale symbols. On the other hand, the positivity condition for S in [14] is only a sufficient condition, with linear formulation in S that does not apply to certain refinable functions (see Remark 8 and Example 4 in Section 5). An advantage of the positivity condition in [14] is that it is easy to apply to the two-scale symbols $((1+z)/2)^m$ of cardinal B-splines, which is discussed in [14], but not completely settled in our paper, except for low order splines and case-by-case verification for higher order ones by using the positivity condition in our paper. Finally, it is worthwhile to point out that our construction procedure only relies on methods of linear algebra and univariate spectral factorization (see Remark 6 in Section 5), whereas other methods usually require solution of a system of quadratic equations, which is often done by Computer Algebra systems such as Singular (www.mathematik.uni-kl.de/~zca/Singular).

2. Notations

Throughout this paper we will consider a compactly supported real-valued refinable function $\phi:\mathbb{R}\to\mathbb{R}$ with finite mask and real mask coefficients; i.e., ϕ satisfies a two-scale relation

$$\phi(x)=\sum_{k=N_1}^{N_2} p_k\phi(2x-k),\quad \text{a.e. } x\in\mathbb{R}, \tag{2.1}$$

for some real numbers p_k. We assume that the corresponding two-scale Laurent polynomial

$$P(z):=\frac{1}{2}\sum_{k=N_1}^{N_2} p_k z^k \tag{2.2}$$

satisfies

$$P(z)=\left(\frac{1+z}{2}\right)^m P_0(z), \tag{2.3a}$$

for some $m\geqslant 1$, with a Laurent polynomial P_0 that satisfies $P_0(-1)\neq 0$. By adopting the definition of Fourier transform

$$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ix\omega}\,\mathrm{d}x,$$

we further require that ϕ satisfies

$$\hat{\phi}(0)=1 \tag{2.3b}$$

and, for convenience, we assume that

$$\phi \text{ is piecewise Lip}\,\alpha, \quad \text{for some } \alpha > 0. \tag{2.3c}$$

Note that Eq. (2.3b) differs from the condition $\hat{\phi}(0) \neq 0$ only by a normalization. The smoothness condition (2.3c) can be further weakened (see, e.g., [7]), and is sufficient to conclude that every finite linear combination of integer translates of ϕ, whose coefficients sum to zero, generates a Bessel sequence (see [7, Theorem 1]). However, stability or Riesz condition for the spanning sets of the nested subspaces

$$V_j := \operatorname{clos}_{L^2} \operatorname{span}\{\phi_{j,k} := 2^{j/2}\phi(2^j \cdot -k)\colon\ k \in \mathbb{Z}\},$$

where $L^2 := L^2(\mathbb{R})$, is not required in this paper. Properties (2.3b) and (2.2) imply respectively the density and trivial intersection property of the nested sequence

$$\{0\} \leftarrow \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \to L^2 \tag{2.4}$$

(see, for example [3] and [4, p. 121]).

With $z = \mathrm{e}^{-i\omega/2}$ on the (complex) unit circle $\mathbb{T}$, Eq. (2.1) is equivalent to

$$\hat{\phi}(\omega) = P(z)\hat{\phi}(\omega/2), \quad \text{a.e. } \omega \in \mathbb{R}. \tag{2.5}$$

We then study two finite families $\{\psi_i\}, \{\tilde{\psi}_i\} \in L^2$, defined by scaling relations

$$\hat{\psi}_i(\omega) = Q_i(z)\hat{\phi}(\omega/2); \qquad \hat{\tilde{\psi}}_i(\omega) = \widetilde{Q}_i(z)\hat{\phi}(\omega/2), \quad i = 1, \ldots, N, \tag{2.6}$$

where Q_i, $\widetilde{Q}_i$ are Laurent polynomials that have real coefficients and vanish at $z = 1$. In other words,

$$Q_i(z) = \left(\frac{1-z}{2}\right)^{m_i} q_i(z), \qquad \widetilde{Q}_i(z) = \left(\frac{1-z}{2}\right)^{\tilde{m}_i} \tilde{q}_i(z),$$

where $m_i, \tilde{m}_i \geqslant 1$. Hence, the functions ψ_i and $\tilde{\psi}_i$ have compact support and at least one vanishing moment.

Our study of affine frames involves the two families of shifts and dilates,

$$\begin{aligned} \Psi &:= \{\psi_{i;\,j,k} = 2^{j/2}\psi(2^j \cdot -k)\colon\ 1 \leqslant i \leqslant N,\ j,k \in \mathbb{Z}\}, \\ \widetilde{\Psi} &:= \{\tilde{\psi}_{i;\,j,k} = 2^{j/2}\tilde{\psi}(2^j \cdot -k)\colon\ 1 \leqslant i \leqslant N,\ j,k \in \mathbb{Z}\}. \end{aligned} \tag{2.7}$$

As mentioned above, condition (2.3c) ensures that both sets are Bessel families in L^2 (see [7, Theorem 1]).

Our objective is the study of Bessel families that satisfy duality relations of the following form.

Definition 1. The two families Ψ and $\widetilde{\Psi}$ in (2.6)–(2.7) are called *sibling frames*, if they are Bessel families and if the duality relation

$$\langle f, g\rangle = \sum_{i=1}^{N} \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{i;\,j,k}\rangle\, \langle \tilde{\psi}_{i;\,j,k}, g\rangle \tag{2.8}$$

is satisfied for all $f, g \in L^2$.

We note that both families are indeed frames of L^2. As usual, the frame condition for Ψ is defined by

$$A\|f\|^2 \leqslant \sum_{i=1}^{N} \sum_{j,k\in\mathbb{Z}} \left|\langle f, \psi_{i;\,j,k}\rangle\right|^2 \leqslant B\|f\|^2, \quad f \in L^2,$$

where A, B are positive constants. The upper frame bound B exists, because Ψ is a Bessel family. The lower frame bound A results from the duality (2.8) with the Bessel family $\widetilde{\Psi}$. For ease of notation, we will call the families $\{\psi_i\}$ and $\{\tilde{\psi}_i\}$ of frame generators sibling frames as well.

Note that sibling frames are generated by functions $\psi_i, \tilde{\psi}_i \in V_1$; i.e., both families are derived from the same refinable function ϕ. Thus, our present development describes a more general framework than orthonormal wavelet bases or tight frames. They also provide much more freedom than the initial definition of MRA-frames by Benedetto and Li [2] where orthogonality between scaling levels was required and where the family $\{\phi(\cdot - k)\}$ was supposed to be a frame of V_0. Sibling frames can also be viewed as biframes (or dual frames, see [1,30]) with the same refinable function. We will show that this new concept gives enough flexibility for the realization of important properties such as symmetry, small support, and a high order of vanishing moments. These can be achieved when using only two generators for each of the two families Ψ and $\widetilde{\Psi}$.

The following result gives a precise characterization of duality of two frames (see [15, 18,19,29,30]). Note that no reference is made to an underlying refinable function.

Theorem A. *If the affine families Ψ and $\widetilde{\Psi}$ are Bessel families, then the duality relation* (2.8) *holds, if and only if*

$$\sum_{j\in\mathbb{Z}}\sum_{i=1}^{N}\hat{\psi}_i(2^j\omega)\overline{\hat{\tilde{\psi}}_i(2^j\omega)} = 1 \tag{2.9}$$

and

$$\sum_{j=0}^{\infty}\sum_{i=1}^{N}\hat{\psi}_i(2^j\omega)\overline{\hat{\tilde{\psi}}_i(2^j(\omega+2k\pi))} = 0 \tag{2.10}$$

a.e. in $\mathbb{R}$, where (2.10) *holds for all odd integers k.*

The assumption on Ψ being a Bessel family is not needed for tight frames.

3. Characterization of sibling frames and VMR functions

The results in this section are extensions of earlier work by Weiss et al. [15,19], Han [18], and Ron and Shen [29,30], who have developed a characterization of tight affine frames and results on dual pairs of affine frames (so-called biframes), in that the wavelet frames are associated with certain multiresolution analysis. Parallel investigations by Daubechies et al. [14] are currently done.

Our first goal is to obtain a complete characterization of sibling frames generated from a compactly supported refinable function. This characterization will be useful for constructions of frames with maximal order of vanishing moments. An essential role is played by a certain parameter function $S(z)$, which can be characterized as the quotient of two Laurent polynomials. This function will provide a tool for the design of frames with vanishing moments, and hence, will be called a *vanishing-moment recovery* (VMR) function.

Theorem 1. *Let ϕ be a refinable function with compact support and two-scale Laurent polynomial symbol P with real coefficients such that* (2.3a)–(2.3c) *are satisfied. Let Q_i and $\widetilde{Q}_i$ be Laurent polynomials with real coefficients vanishing at $z=1$. Then the functions ψ_i and $\tilde{\psi}_i$ defined in* (2.6) *generate sibling frames of L^2, if and only if there exists a VMR function S, defined a.e. in $\mathbb{C}$, that satisfies the following properties*:

- (i) *S is the quotient of two Laurent polynomials with real coefficients, $S(z) = R(z)/T(z)$*;
- (ii) *S is continuous at $z=1$, and $S(1)=1$*;
- (iii) *for almost all $z\in\mathbb{T}$ the following two equations hold*:

$$S(z^2)|P(z)|^2 + \sum_i Q_i(z)\overline{\widetilde{Q}_i(z)} = S(z), \tag{3.1}$$

$$S(z^2)P(z)\overline{P(-z)} + \sum_i Q_i(z)\overline{\widetilde{Q}_i(-z)} = 0. \tag{3.2}$$

As pointed out in the introduction, the fundamental function

$$\Theta(\omega) := \frac{1}{|\hat{\phi}(\omega)|^2} \sum_{j=1}^{\infty} \sum_{i=1}^{N} \hat{\psi}_i(2^j\omega)\overline{\hat{\tilde{\psi}}_i(2^j\omega)}$$
$$= \sum_{j=1}^{\infty} \sum_{i=1}^{N} Q_i(z^{2^j})\overline{\widetilde{Q}_i(z^{2^j})} \prod_{k=1}^{j-1} \left|P(z^{2^k})\right|^2 \tag{3.3}$$

in [29], defined in terms of both P and Q_i, $\widetilde{Q}_i$, agrees with the VMR function $S(z)$, where $z = \mathrm{e}^{-i\omega/2}$, and hence satisfies (3.1). Our point of view in the following sections is that the VMR function S should be defined independently of Q_i and $\widetilde{Q}_i$, and Eqs. (3.1) and (3.2) provide a vehicle for finding Q_i and $\widetilde{Q}_i$.

Proof. We borrow from and extend an idea in [29] by defining the auxiliary function $\Theta(\omega)$ in (3.3). Since both families are Bessel (see [7, Theorem 1]), the series in the first line of (3.3) converges absolutely almost everywhere. Furthermore, $\hat{\phi}$ is nonzero almost everywhere due to analyticity. Hence, Θ is a measurable function, which can be defined by the series of Laurent polynomials in the second line of (3.3). This shows that Θ is 2π-periodic.

We use this function when employing the characterizing equations of Theorem A. Eq. (2.9) gives

$$1 = \sum_{j\in\mathbb{Z}} \sum_{i} \hat{\psi}_i(2^j\omega)\overline{\hat{\tilde{\psi}}_i(2^j\omega)} = \lim_{J\to-\infty} \sum_{j=J}^{\infty} \sum_{i} \hat{\psi}_i(2^j\omega)\overline{\hat{\tilde{\psi}}_i(2^j\omega)}$$
$$= \lim_{J\to-\infty} \left[\Theta(2^J\omega)\left|\hat{\phi}(2^{J-1}\omega)\right|^2\right].$$

The continuity (in fact, analyticity) of $\hat{\phi}$ at $\omega = 0$ shows that Eq. (2.9) is equivalent to the relation

$$\lim_{J\to-\infty} \Theta(2^J\omega) = 1 \tag{3.4}$$

for almost every $\omega \in \mathbb{R}$. In a similar manner, we obtain an equivalent representation of equation (2.10) by considering

$$0 = \sum_{j\geqslant 0} \sum_{i} \hat{\psi}_i(2^j\omega)\overline{\hat{\tilde{\psi}}_i(2^j(\omega + 2k\pi))}$$
$$= \hat{\phi}(\omega/2)\overline{\hat{\phi}(\omega/2 + k\pi)}\left[\sum_{i} Q_i(z)\overline{\widetilde{Q}_i(-z)} + P(z)\overline{P(-z)}\Theta(\omega)\right],$$

for almost every $\omega \in \mathbb{R}$ and every odd integer k. Then, the analyticity of $\hat{\phi}$ leads to

$$0 = \sum_{i\in I} Q_i(z)\overline{\widetilde{Q}_i(-z)} + P(z)\overline{P(-z)}\Theta(\omega), \quad \text{a.e. } z \in \mathbb{T}. \tag{3.5}$$

Let us now prove both directions of the equivalence in Theorem 1. First we assume that the families $\{\psi_i\}$ and $\{\tilde{\psi}_i\}$ satisfy the duality relation (2.8). We show that the function $S(z) = \Theta(\omega/2)$, where $z = \mathrm{e}^{-i\omega/2}$, has the properties (i)–(iii) stated in Theorem 1. As a result of (3.5), we have

$$P(z)\overline{P(-z)}S(z^2) = -\sum_{i\in I} Q_i(z)\overline{\widetilde{Q}_i(-z)}, \quad |z| = 1.$$

Since the coefficients of all two-scale symbols are real, we can infer that

$$\overline{P(-z)} = P(-1/z) \quad \text{and} \quad \overline{\widetilde{Q}_i(-z)} = \widetilde{Q}_i(-1/z).$$

Hence, $S(z^2) = X(z)/Y(z)$ is a quotient of two Laurent polynomials with real coefficients. Rewriting $S(z^2)$ as an average $(X(z)/Y(z) + X(-z)/Y(-z))/2$ gives a representation of the form

$$S(z^2) = R(z^2)/T(z^2),$$

thus showing property (i). Property (ii) follows from Eq. (3.4) and the fact that S, as a rational function in $\mathbb{C}$, is continuous everywhere except for a finite number of singularities. The second relation of property (iii) was already established above, while the first relation in (iii) is an immediate consequence of (3.3).

Let us now assume that there is a rational function S that satisfies conditions (i)–(iii) in the theorem. We will derive Eqs. (2.9) and (2.10) from here. Note that the first equation in (iii) implies, by multiplication of both sides by $|\hat{\phi}(\omega/2)|^2$, that

$$S(z)\left|\hat{\phi}(\omega/2)\right|^2 = S(z^2)\left|\hat{\phi}(\omega)\right|^2 + \sum_i \hat{\psi}_i(\omega)\overline{\hat{\tilde{\psi}}_i(\omega)}.$$

For the proof of (2.9), we apply the above relation recursively, and obtain for any $r < s$ in $\mathbb{Z}$ that

$$S\bigl(z^{2^r}\bigr)\left|\hat{\phi}\bigl(2^{r-1}\omega\bigr)\right|^2 = \sum_{j=r}^{s-1}\sum_i \hat{\psi}_i\bigl(2^j\omega\bigr)\overline{\hat{\tilde{\psi}}_i\bigl(2^j\omega\bigr)} + S\bigl(z^{2^s}\bigr)\left|\hat{\phi}\bigl(2^{s-1}\omega\bigr)\right|^2.$$

Property (ii) and the continuity of $\hat{\phi}$ at zero assure that the limit on the left-hand side is 1, as r tends to $-\infty$. Furthermore, we already know that the series on the right-hand side converges absolutely a.e., by the assumption that both families are Bessel. Hence, we can conclude that

$$\lim_{s\to\infty} S\bigl(z^{2^s}\bigr)\left|\hat{\phi}\bigl(2^{s-1}\omega\bigr)\right|^2 = a(\omega)$$

exists a.e. Now, assume, on the contrary, that this limit is nonzero on a set E of positive measure. Then appealing to the Riemann–Lebesgue Theorem for the function ϕ necessitates the condition

$$\lim_{s\to\infty}\left|S\bigl(z^{2^s}\bigr)\right| = \infty$$

for all $z = \mathrm{e}^{-i\omega/2}$, $\omega \in E$. But this is impossible for a rational function S. We have thus established that $a(\omega) = 0$, and therefore (2.9) is valid.

Similarly, we find, from the second equation of property (iii), that for any odd integer k and any $s > 0$,

$$0 = \sum_{j=0}^{s-1}\sum_i \hat{\psi}_i\bigl(2^j\omega\bigr)\overline{\hat{\tilde{\psi}}_i\bigl(2^j(\omega+2k\pi)\bigr)} + S\bigl(z^{2^s}\bigr)\hat{\phi}\bigl(2^{s-1}\omega\bigr)\overline{\hat{\phi}\bigl(2^{s-1}(\omega+2k\pi)\bigr)}.$$

The same considerations as above lead to

$$\lim_{s\to\infty} S\bigl(z^{2^s}\bigr)\hat{\phi}\bigl(2^{s-1}\omega\bigr)\overline{\hat{\phi}\bigl(2^{s-1}(\omega+2k\pi)\bigr)} = 0,$$

and this gives Eq. (2.10). Thus, we have shown that Ψ and $\widetilde{\Psi}$ satisfy the duality relation (2.8). □

Remark 1. Existing constructions of tight frames in the literature only consider the special VMR function $S \equiv 1$ in conditions (3.1) and (3.2). Theorem 1 shows that these conditions (with $S \equiv 1$ and $\widetilde{Q}_i = Q_i$) are sufficient, but not necessary for the construction of tight frames. A different proof for the sufficiency in this special case can be derived by using a telescoping argument applied to

$$\sum_{k\in\mathbb{Z}}\left|\langle f, \phi_{j+1,k}\rangle\right|^2 = \sum_{k\in\mathbb{Z}}\left|\langle f, \phi_{j,k}\rangle\right|^2 + \sum_i\sum_{k\in\mathbb{Z}}\left|\langle f, \psi_{i;j+1,k}\rangle\right|^2,$$

which follows from (3.1) and (3.2), see [6,12, p. 264]. Limits of the series on the left-hand side for $j \to \pm\infty$ exist by virtue of the assumptions (2.3a)–(2.3c). There is a straightforward generalization of this method of proof to Laurent polynomial VMR functions S and sibling frames. We were unable, however, to use the same argument for rational VMR functions S.

It will be useful to draw stronger conclusions about the function S. Typically we will use Laurent polynomials S in our constructions. The possibility of non-polynomial S is rather restricted, as we will see next. For this purpose we recall some notation from univariate wavelet theory. A set $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{T}$, $n \geqslant 2$, of distinct complex numbers is a nontrivial "cycle" if $z_k = z_{k-1}^2$ for $2 \leqslant k \leqslant n$ and $z_1 = z_n^2$. Cycles play an important role in characterizing stability of integer translates of a refinable function ϕ, typically denoted as Cohen's condition (see [11,17]). The following result was obtained in [11].

Theorem B. *Let P be a Laurent polynomial that satisfies $P(1) = 1$, $P(-1) = 0$, for which no pair of symmetric roots of P (i.e., $P(z) = P(-z) = 0$) exists on $\mathbb{T}$, and that the associated scaling function ϕ is in L^2. Then the integer shifts of ϕ are stable if and only if there exists no nontrivial cycle $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{T}$ so that $P(-z_k) = 0$ for $1 \leqslant k \leqslant n$.*

Thus, it follows from the following result that non-polynomial S can only occur when the integer shifts of ϕ are not stable. In this case, the denominator T of the rational VMR function S can be further analyzed.

Proposition 1. *With the same notations as in Theorem* 1, *let $S = R/T$ be the quotient of two Laurent polynomials with real coefficients that have no common roots in $\mathbb{C} \setminus \{0\}$. Assume that S satisfies properties* (i)–(iii) *in Theorem* 1. *Let $\mathcal{Z}(T)$ denote the set of complex roots of T different from* 0. *Then the following statements hold*:

(a) $\mathcal{Z}(T) \subset \mathbb{T} \setminus \{z_{j,\ell} := \mathrm{e}^{i2\pi\ell/2^j}\colon\ 0 \leqslant \ell < 2^j,\ j \geqslant 0\}$.
(b) *If $\mathcal{Z}(T)$ is non-empty, then it contains at least one non-trivial cycle.*
(c) *If $\mathcal{Z}(T)$ is non-empty, and $P(z)$ and $P(-z)$ have no common roots on $\mathbb{T}$, then $\mathcal{Z}(T)$ is the union of a finite number of non-trivial cycles and each root in a given cycle has the same multiplicity.*
(d) *$P(-w) = 0$ for all $w \in \mathcal{Z}(T)$. In particular, if the integer shifts of ϕ are stable, then $\mathcal{Z}(T)$ is empty, and S must be a Laurent polynomial.*

Proof. Let $S = R/T$ be given as described in the proposition, and assume that T is not a monomial, so that $\mathcal{Z}(T)$ is non-empty. For $w \in \mathcal{Z}(T)$, we introduce the notation

$$E_w := \{w^{2^k}\colon\ k \geqslant 0\}.$$

By using the fact that P, Q_i, and $\widetilde{Q}_i$ are Laurent polynomials with real coefficients, Eqs. (3.1) and (3.2) can be written as

$$\frac{R(z^2)}{T(z^2)} P(z)P(1/z) + \sum_i Q_i(z)\widetilde{Q}_i(1/z) = \frac{R(z)}{T(z)}, \tag{3.6}$$

$$\frac{R(z^2)}{T(z^2)} P(z)P(-1/z) + \sum_i Q_i(z)\widetilde{Q}_i(-1/z) = 0, \tag{3.7}$$

for all $z \in \mathbb{C} \setminus \{0\}$. The root $w \in \mathcal{Z}(T)$ defines a pole of the function on the right-hand side of Eq. (3.6). Consequently, w^2 must be a root of T as well. By repeating this argument, we can show that all elements of E_w are roots of T. Since T is a Laurent polynomial, E_w must be finite. Hence, w must lie on the unit circle. Since $S(1) = 1$, we know that $1 \notin E_w$. This is enough for establishing parts (a) and (b) of the proposition; indeed, w cannot be any of the numbers $z_{j,\ell}$ in part (a), and E_w contains a nontrivial cycle.

We next prove part (d) of the proposition. If we insert Eq. (3.7) into (3.6), we obtain

$$\frac{R(z)}{T(z)} = \sum_i Q_i(z)\widetilde{Q}_i(1/z) - \frac{P(1/z)}{P(-1/z)} \sum_i Q_i(z)\widetilde{Q}_i(-1/z).$$

This implies that $P(-1/w) = 0$ for all $w \in \mathcal{Z}(T)$. On the other hand, we know from part (a) that all of the roots lie on $\mathbb{T}$. Hence, we obtain $0 = \overline{P(-1/w)} = P(-1/\overline{w}) = P(-w)$ for every $w \in \mathcal{Z}(T)$. Together with part (b), we have thus shown that P cannot satisfy Cohen's criterion unless $\mathcal{Z}(T)$ is empty.

Finally, we assume, in addition, that $P(z)$ and $P(-z)$ have no common roots on $\mathbb{T}$. We will show that for every $w \in \mathcal{Z}(T)$ the set E_w is a non-trivial cycle, and that every root $y \in E_w$ has the same multiplicity. (For part (a), we only showed that E_w contains a non-trivial cycle, which means that there are integers $m > n \geqslant 0$ so that $w^{2^m} = w^{2^n}$ and $w^{2^{n+1}} \neq w^{2^n}$.) Let $u \in \mathbb{T}$ be given so that $w = u^2 \in \mathcal{Z}(T)$ is a root of multiplicity m. Then we obtain

$$T(z^2) = (z^2 - w)^m T_0(z^2) = (z-u)^m (z+u)^m T_0(z^2),$$

with a Laurent polynomial T_0 that does not vanish at w. By assumption, either $P(u) \neq 0$ or $P(-u) \neq 0$. If $P(u) \neq 0$, then $P(1/u) = \overline{P(u)} \neq 0$ as well, and the common multiplicity of poles in Eq. (3.6) implies that

$$T(z) = (z-u)^m \widetilde{T}_0(z), \quad \widetilde{T}_0(u) \neq 0.$$

Alternatively, if $P(-u) \neq 0$, we obtain

$$T(z) = (z+u)^m \widetilde{T}_0(z), \quad \widetilde{T}_0(-u) \neq 0.$$

Thus, we have shown that if $w = u^2$ is a root of T of multiplicity m, then either u or $-u$ is a root with the same multiplicity. This enables us to define a set

$$F_w := \{w = w_0, w_1, w_2, \ldots\} \subset \mathcal{Z}(T),$$

where each w_k is a root of T with the same multiplicity m, and $w_k^{2^k} = w$. Finiteness of F_w implies that this set is a non-trivial cycle that contains $w = w_0$. This gives $w = w_k$ for some $k > 0$, and it follows immediately that $F_w = E_w$. This completes the proof of Proposition 1. □

In the following sections we will employ the VMR function S as a means to construct sibling frames with certain desirable properties. In most practical examples, we restrict ourselves to the use of Laurent polynomials S. The previous result shows that this is not a restriction at all, if we deal with compactly supported scaling functions whose integer shifts are stable.

For later use, we state another simplification of the rational Laurent polynomial S.

Lemma 1. *Let T be a Laurent polynomial with real coefficients, whose roots lie on $\mathbb{T} \setminus \{-1, 1\}$. Then T has the form*

$$T(z) = t_0 z^\ell T_0(z), \tag{3.8}$$

where $t_0 \in \mathbb{R}$, ℓ is an integer, and T_0 is a Laurent polynomial that is real on $\mathbb{T}$ and has real coefficients.

Proof. All roots of T are pairs of complex conjugate numbers w and $\bar{w} = 1/w$. Therefore, T has a representation

$$T(z) = t_0 z^\ell \prod_{j=1}^{r} \bigl(z + 1/z - 2\,\mathrm{Re}(w_k)\bigr) = t_0 z^\ell T_0(z),$$

where $t_0 \in \mathbb{R}$ and ℓ is an integer. $T_0(z)$ has real coefficients and is real on $\mathbb{T}$. □

4. Sibling frames with two generators

It was observed by several authors that the construction of tight affine frames based on "unitary matrix extension" [29] has certain restrictions. For example, in [6] it was pointed out that the method can be used only if

$$|P(z)|^2 + |P(-z)|^2 \leqslant 1, \quad z \in \mathbb{T}. \tag{4.1}$$

Furthermore, the construction of frames from B-spline multiresolution using unitary matrix extension necessarily leads to frames where at least one generator ψ_i has only one vanishing moment. Indeed, if the Laurent polynomial symbols Q_i satisfy

$$\sum_i \left|Q_i(z)\right|^2 = 1 - \left|P(z)\right|^2, \quad z \in \mathbb{T},$$

where $P(z) = 2^{-m}(1+z)^m$ is the two-scale Laurent polynomial symbol of the cardinal B-spline of order m, then the highest power of $(1-z)$ that can be factored out on both sides of this equation is $(1-z)^2$.

In this section we present a method that makes use of the VMR function S in Theorem 1 for the design of new sibling frames Ψ, $\widetilde{\Psi}$. This method neither underlies restriction (4.1), nor imposes restrictions on the order of vanishing moments of ψ_i and $\tilde{\psi}_i$, other than the order of $z=-1$ as a root of the Laurent polynomial P. Further properties such as orthogonality between spaces generated by integer translates of each of ψ_1 and ψ_2 and construction schemes of tight frames will be studied in the remaining sections of this article.

Our main concern is the study of sibling frames with two generators, namely: $\Psi = \{\psi_1, \psi_2\}$, $\widetilde{\Psi} = \{\tilde{\psi}_1, \tilde{\psi}_2\}$. Certain negative results on existence of sibling frames with only one generator will be given in the last section.

The important identities in Theorem 1, part (iii), can be stated as

$$\begin{aligned}\mathcal{M}(z) &:= \begin{bmatrix} S(z) - S(z^2)P(z)P(1/z) & -S(z^2)P(1/z)P(-z) \\ -S(z^2)P(z)P(-1/z) & S(-z) - S(z^2)P(-z)P(-1/z) \end{bmatrix} \\ &= \sum_i \begin{bmatrix} \widetilde{Q}_i(1/z) \\ \widetilde{Q}_i(-1/z) \end{bmatrix} \begin{bmatrix} Q_i(z) & Q_i(-z) \end{bmatrix}, \end{aligned} \tag{4.2}$$

where S is the VMR function described in Theorem 1. The last identity can be rewritten as a matrix factorization, which in the case of only two generators takes on the form

$$\mathcal{M}(z) = \begin{bmatrix} \widetilde{Q}_1(1/z) & \widetilde{Q}_2(1/z) \\ \widetilde{Q}_1(-1/z) & \widetilde{Q}_2(-1/z) \end{bmatrix} \begin{bmatrix} Q_1(z) & Q_1(-z) \\ Q_2(z) & Q_2(-z) \end{bmatrix}. \tag{4.3}$$

The essential step consists of defining such a function S which is a Laurent polynomial or the quotient of two Laurent polynomials with real coefficients, such that $S(1) = 1$ and

$$S(z) - \frac{1}{B_\phi(z)} = \mathcal{O}\big(|z-1|^{2m}\big) \quad \text{near } z = 1. \tag{4.4}$$

Here, B_ϕ denotes the generalized Euler–Frobenius polynomial associated with the refinable function ϕ defined by

$$B_\phi(z) = \sum_{k \in \mathbb{Z}} b_k z^k, \quad \text{where } b_k = \int_{\mathbb{R}} \phi(x)\phi(x+k)\,\mathrm{d}x.$$

Let us recall from [4, Theorem 5.10] that B_ϕ is a Laurent polynomial with real coefficients, non-negative on $\mathbb{T}$, and $B_\phi(1) = 1$, and that the relation

$$P(z)P(1/z)B_\phi(z) + P(-z)P(-1/z)B_\phi(-z) = B_\phi\big(z^2\big) \tag{4.5}$$

holds for all complex $z \neq 0$.

We will see in Section 4.1 that (4.4) governs the vanishing-moment recovery property of frames. Note that property (ii) in Theorem 1, namely that S is continuous at 1 and $S(1) = 1$, is a direct consequence of (4.4). There are many ways to define a Laurent polynomial S that satisfies (4.4). One particular choice is the Taylor polynomial of degree $2k-1$ of $1/B_\phi$, with center $z_0 = 1$. Another, more symmetric, choice is

$$S(z) = \sum_{k=0}^{m-1} s_k (2 - z - 1/z)^k,$$

where the real coefficients s_k are determined by a linear system of equations. Consistency of this system is assured by the fact that B_ϕ has an expansion in powers of $(2 - z - 1/z)$, due to the symmetry relation $b_k = b_{-k}$ for its coefficient sequence.

4.1. Vanishing moments

Our main result in this section is that there always exist sibling frames with two generators and with the maximal number of vanishing moments. Moreover, these frames can be chosen to be symmetric or antisymmetric as governed by the order of the root $z = -1$ of the two-scale symbol of ϕ, provided that ϕ is symmetric.

Theorem 2. *For any compactly supported refinable function ϕ that satisfies* (2.3a)–(2.3c), *there exist compactly supported sibling frames $\{\psi_1, \psi_2\}$, $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ with the property that all of the four functions have m vanishing moments, where m is the order of the root $z = -1$ of the two-scale Laurent polynomial P. Furthermore, if ϕ is symmetric, then all of the four functions can be chosen to be symmetric for even m, and antisymmetric for odd m.*

Proof. Our proof is constructive. Since it is similar to the proof of an independent but earlier result in [13], we only give an outline in the following. (We thank one of the reviewers for pointing out the reference [13] which allows us to shorten our original presentation.) We can choose S to be any Laurent polynomial that has real coefficients and satisfies property (4.4). Let V be a neighborhood of $z = 1$ where $B_\phi(z)$ and $B_\phi(z^2)$ are non-zero. For all $z \in V$, we infer from (4.4) and (4.5) that

$$S(z) - S(z^2)P(z)P(1/z) = \frac{P(-z)P(-1/z)B_\phi(-z)}{B_\phi(z)B_\phi(z^2)} + \mathcal{O}(|z-1|^{2m})$$
$$= \mathcal{O}(|z-1|^{2m}).$$

Hence, the matrix (4.2) can be factored in the form of

$$\mathcal{M}(z) = D_m(1/z)\begin{bmatrix} A(z) & -S(z^2)P_0(1/z)P_0(-z) \\ -S(z^2)P_0(z)P_0(-1/z) & A(-z) \end{bmatrix} D_m(z), \quad (4.6)$$

where D_m is the diagonal matrix $D_m(z) := \operatorname{diag}(((1-z)/2)^m, ((1+z)/2)^m)$ and A is some symmetric Laurent polynomial with real coefficients. The matrix relation (4.3) can be satisfied by taking

$$\begin{aligned} Q_1(z) &= ((1-z)/2)^m (A(z) - S(z^2)P_0(z)P_0(-1/z))/2, \\ \widetilde{Q}_1(z) &= ((1-z)/2)^m, \\ Q_2(z) &= z((1-z)/2)^m (A(z) + S(z^2)P_0(z)P_0(-1/z))/2, \\ \widetilde{Q}_2(z) &= z((1-z)/2)^m. \end{aligned} \quad (4.7)$$

This completes the first part of the proof of Theorem 2.

Observe that $P_0(z)P_0(-1/z) = P_0(1/z)P_0(-z)$, provided that P is symmetric. Hence, Q_1 and Q_2 in (4.7) are symmetric (respectively, antisymmetric) about $m/2$ and $m/2+1$, respectively, if m is even (respectively, odd). Symmetry or antisymmetry of the coefficient sequences of $\widetilde{Q}_1$ and $\widetilde{Q}_2$ is obvious. These symmetry properties of the Laurent polynomials directly relate to the analogous symmetry properties of the functions ψ_1, ψ_2 and $\tilde{\psi}_1$, $\tilde{\psi}_2$. □

Remark 2. It is not difficult to show that the maximal number of vanishing moments of sibling frames cannot exceed m, where m is the order of the root $z = -1$ of the two-scale symbol P in (2.3a). In other words, there is at least one function ψ_i and one corresponding function $\tilde{\psi}_i$ that have at most m vanishing moments. Indeed, Eq. (3.2) gives

$$\sum_i Q_i(z)\overline{\widetilde{Q}_i(-z)} = -S(z^2)P(z)\overline{P(-z)}$$
$$= -\left(\frac{1-\bar{z}}{2}\right)^m \left[S(1)P(1)P_0(-1) + \mathcal{O}(|z-1|^{m+1})\right]$$

by applying (2.3a) and considering the Taylor expansion around $z = 1$. The first term inside the brackets is non-zero. This shows that not all Q_i can have zeros of order greater than

m at $z=1$. In other words, at least one ψ_i has at most m vanishing moments. The same method, using Taylor expansion of the same term around $z=-1$, shows that at least one $\tilde{\psi}_i$ has at most m vanishing moments.

Remark 3. The pair of sibling frames constructed in (4.7) results from a trivial factorization

$$\mathcal{M}_0(z)=\begin{bmatrix}1 & 1/z\\ 1 & -1/z\end{bmatrix}\begin{bmatrix}q_1(z) & q_1(-z)\\ zq_2(z) & -zq_2(-z)\end{bmatrix}, \tag{4.8}$$

where $q_{1,2}(z)=(A(z)\pm S(z^2)P_0(z)P_0(-1/z))/2$, of the reduced matrix

$$\mathcal{M}_0(z)=\begin{bmatrix}A(z) & -S(z^2)P_0(1/z)P_0(-z)\\ -S(z^2)P_0(z)P_0(-1/z) & A(-z)\end{bmatrix}, \tag{4.9}$$

which is obtained after cancellation of the factors $D_m(z)$ and $D_m(1/z)$ in (4.6). The resulting frame generators $\{\tilde{\psi}_1,\tilde{\psi}_2\}$ are chosen to satisfy

$$\tilde{\psi}_2(x)=\tilde{\psi}_1(x-1/2).$$

This is significant in that the shift-variant defect of the standard wavelet decomposition procedure (of discrete convolution followed by down-sampling) can be eliminated. In a different context, Kingsbury [20,21] considered a dual tree of wavelet filters, where all the sampling rates of the fully decimated wavelet transform are doubled by eliminating the downsampling operation in the first decomposition step and where the filters of the subsequent decomposition steps are chosen with alternating parity, in order to achieve the almost shift-invariant effect, with noticeable improvement in image denoising and texture analysis. In Fig. 1, we show graphs of the four generators of the pair of sibling frames with four vanishing moments that are linear combinations of cubic cardinal B-splines. The trivial factorization (4.8) is employed here. The aforementioned exact shift invariance of $\{\tilde{\psi}_1,\tilde{\psi}_2\}$ can be recognized in Fig. 1b. Approximate shift-invariance of their dual $\{\psi_1,\psi_2\}$ can be seen in Fig. 1a.

Remark 4. The construction based on (4.8) may result in unbalanced supports for the two generators and their duals: the length of the coefficient sequences of $\widetilde{Q}_i$, $i=1,2$, is $m+1$, while the length of Q_i is $m+\ell_i$ where ℓ_i is the length of the coefficient sequence of q_i in (4.8). More "balanced" factorizations

$$\mathcal{M}_0(z)=\begin{bmatrix}\tilde{q}_1(1/z) & \tilde{q}_2(1/z)\\ \tilde{q}_1(-1/z) & \tilde{q}_2(-1/z)\end{bmatrix}\begin{bmatrix}q_1(z) & q_1(-z)\\ q_2(z) & q_2(-z)\end{bmatrix} \tag{4.10}$$

can be constructed for special cases where the determinant of the matrix $\mathcal{M}_0$ has low degree. This can occur, of course, even if the entries of $\mathcal{M}_0$ have higher degree. As a rule of thumb, we will obtain a factorization where all coefficient sequences have half the length of the sequences q_1 and q_2 in the trivial factorization (4.8). Ingredients of our construction are a polyphase decomposition and degree reduction using the Euclidean algorithm. A precise description of this result is given in the appendix. The following examples can serve as an explanation of this method.

Example 1. The refinable function ϕ is chosen to be the cardinal B-spline N_m of order m (or degree $m-1$) with integer knots, and supported on the interval $[0,m]$. Its two-scale symbol is $P(z)=((1+z)/2)^m$. We choose the vanishing-moment recovery function $S(z)$ to be the symmetric Laurent polynomial

$$S(z)=\sum_{j=0}^{m-1}s_j\left(\frac{2-z-z^{-1}}{4}\right)^j$$

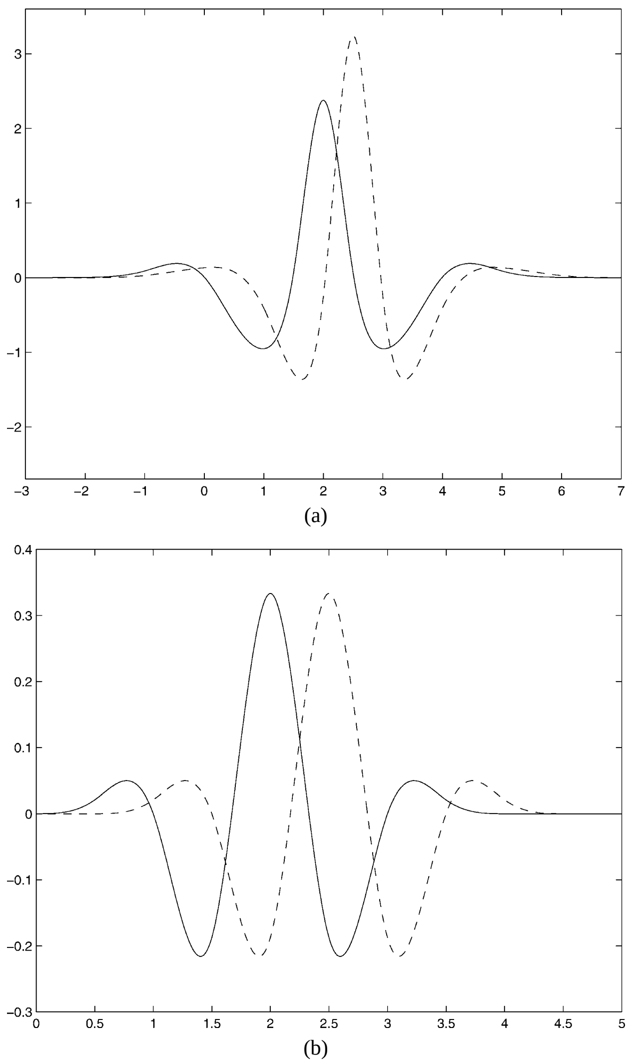


Fig. 1. Cubic spline sibling frames $\{\psi_1, \psi_2\}$ (a) and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ (b); exact shift-invariance for $\tilde{\psi}_2 = \tilde{\psi}_1(\cdot - 1/2)$.

of lowest degree for which (4.4) is satisfied. An explicit form of the coefficients s_k is given by

$$s_0 = 1, \qquad s_k = \frac{1}{4^k - 1} \sum_{\ell=0}^{k-1} (-1)^{k-1-\ell} 4^\ell s_\ell \binom{m+\ell}{k-\ell}. \tag{4.11}$$

For each $m = 2, 3, 4$, we may easily write down S and the factorization of the matrix $\mathcal{M}$ in (4.3) which defines either a tight frame, or a sibling frame. Details for obtaining these factorizations are given in Appendix A.

(i) For the linear cardinal B-spline $N_2(x) = (1 - |x-1|)_+$, the vanishing-moment recovery function $S(z) = 1 + \frac{1}{6}(2 - z - z^{-1})$ reveals two vanishing moments. We obtain a tight frame with two symmetric generators ψ_1 and ψ_2 from the factorization $\mathcal{M}(z) = \mathcal{Q}(z^{-1})\mathcal{Q}^{\mathrm{T}}(z)$, where

$$\begin{aligned}\mathcal{Q}(z) &= \begin{bmatrix} \left(\frac{1-z}{2}\right)^2 & 0 \\ 0 & \left(\frac{1+z}{2}\right)^2 \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \begin{bmatrix} 1 & \frac{1+z^2}{4} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{8/3} \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{1-z}{2}\right)^2 & 0 \\ 0 & \left(\frac{1+z}{2}\right)^2 \end{bmatrix} \begin{bmatrix} 1 & \left(1+4z+z^2\right)/\sqrt{6} \\ 1 & \left(1-4z+z^2\right)/\sqrt{6} \end{bmatrix}.\end{aligned}$$

Hence, the two-scale symbols for ψ_1, ψ_2 are given by

$$Q_1(z) = \left((1-z)/2\right)^2, \qquad Q_2(z) = \left((1-z)/2\right)^2\left(1+4z+z^2\right)/\sqrt{6}.$$

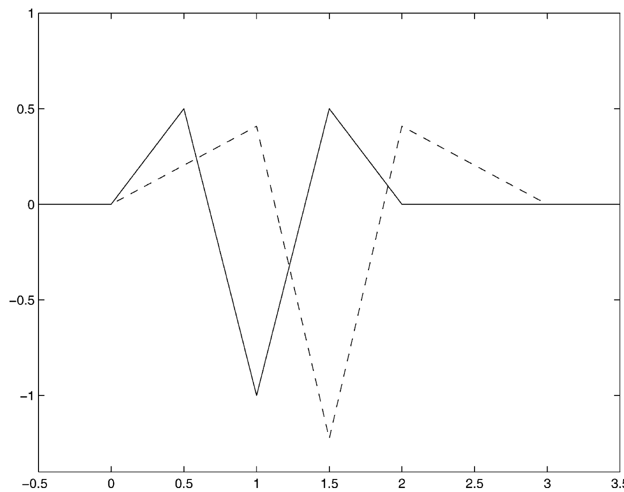


Fig. 2. Linear spline tight frame generators ψ_1 (solid line) and ψ_2 (dashed line) with two vanishing moments.

The graphs are shown in Fig. 2. Note that the construction in [29] gives a tight frame with two generators, one symmetric and the other antisymmetric, where the symmetric generator has only one vanishing moment.

(ii) The construction for quadratic cardinal B-spline N_3 makes use of $S(z) = 1 + \frac{1}{4}(2 - z - z^{-1}) + \frac{13}{240}(2 - z - z^{-1})^2$ to reveal three vanishing moments. The factorization $\mathcal{M}(z) = \mathcal{R}(z^{-1})\mathcal{D}(z)\mathcal{R}^{\mathrm{T}}(z)$ can be formulated with

$$\mathcal{R}(z) = \begin{bmatrix} \left(\frac{1-z}{2}\right)^3 & 0 \\ 0 & \left(\frac{1+z}{2}\right)^3 \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \begin{bmatrix} 1 & \frac{1+z^2}{6} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{52(1+z^{-2})}{103} & 1 \end{bmatrix} \quad \text{and}$$

$$\mathcal{D}(z) = \begin{bmatrix} \frac{103}{15} & 0 \\ 0 & \frac{5438+247(z^2+z^{-2})}{1236} \end{bmatrix}.$$

In order to obtain a tight frame (ψ_1, ψ_2), the diagonal matrix $\mathcal{D}$ is factored out by applying the Riesz Lemma for the second diagonal entry. The factorization

$$5438 + 247(z^2 + z^{-2}) = \tfrac{247}{\lambda}(\lambda + z^2)(\lambda + z^{-2}),$$
$$\lambda = \frac{2719 + 4\sqrt{458247}}{247} \approx 22,$$

gives

$$Q_1(z) = \frac{1}{3\sqrt{1545}}\left(\frac{1-z}{2}\right)^3 \left(361 + 156(z + z^{-1}) + 26(z^2 + z^{-2})\right),$$
$$Q_2(z) = \frac{1}{12}\sqrt{\frac{247}{309\lambda}}\left(\frac{1-z}{2}\right)^3 \left(z^{-2} + 6z^{-1} + (1+\lambda) + 6\lambda z + \lambda z^2\right).$$

It can be seen immediately that ψ_1 is antisymmetric, but ψ_2 is neither symmetric nor antisymmetric (see Fig. 3). Several other choices of frame generators can be made. If symmetry is of no concern, tight frame generators with shorter masks can be found by multiplication of the vector (Q_1, Q_2) by an orthogonal matrix that eliminates two coefficients in either Q_1 or Q_2 with highest (or lowest) powers of z. A construction of frame elements with such short masks (6-tap and 8-tap) was first considered in [14]. Conversely, if symmetry or antisymmetry of both generators is required, a pair of sibling frames $\{\psi_1, \psi_2\}$ and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ of antisymmetric functions can be defined, where $\psi_1 = \tilde{\psi}_1$ is as above, and ψ_2, $\tilde{\psi}_2$ have two-scale symbols

$$Q_2(z) = \frac{1}{6}\sqrt{\frac{2719}{618}}\left(\frac{1-z}{2}\right)^3 (1 + 6z + z^2),$$
$$\widetilde{Q}_2(z) = \left(1 + \frac{247}{5438}(z^2 + z^{-2})\right) Q_2(z).$$

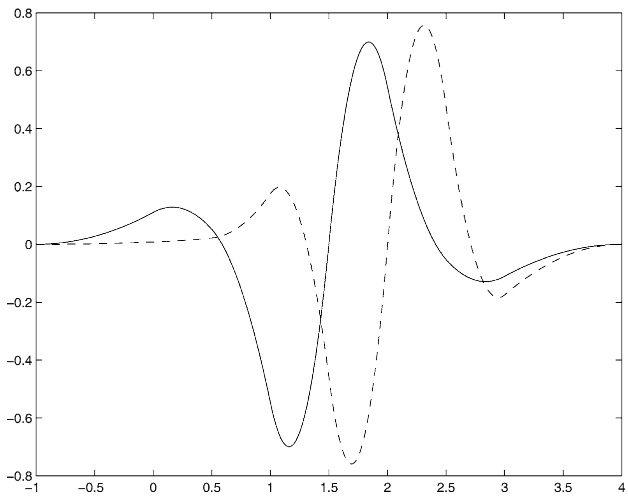


Fig. 3. Quadratic spline tight frame generators ψ_1 (solid line) and ψ_2 (dashed line) with three vanishing moments.

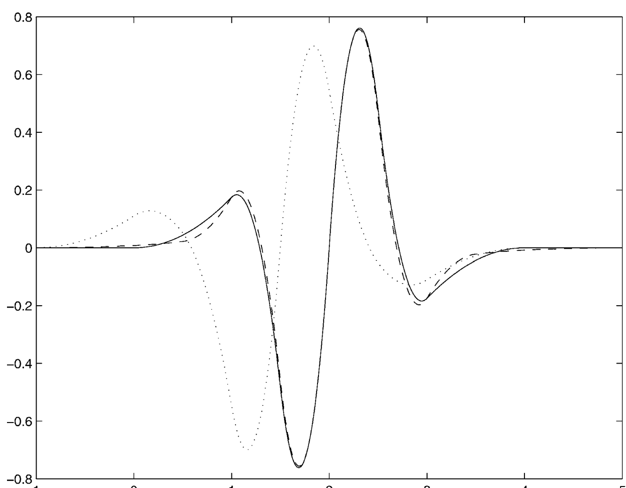


Fig. 4. Quadratic spline sibling frame generators $\psi_1 = \tilde{\psi}_1$ (dotted line), ψ_2 (solid line), and $\tilde{\psi}_2$ (dashed line), with three vanishing moments.

These generators are shown in Fig. 4, where the dotted line depicts the generator $\psi_1 = \tilde{\psi}_1$, the solid line shows ψ_2, and the dashed line depicts $\tilde{\psi}_2$. The supports are $\operatorname{supp}\psi_1 = [-1, 4]$, $\operatorname{supp}\psi_2 = [0, 4]$, and $\operatorname{supp}\tilde{\psi}_2 = [-1, 5]$. Note that $\tilde{\psi}_2$ is a linear combination of integer shifts of ψ_2. The graphs of ψ_2 and $\tilde{\psi}_2$ look almost identical. Furthermore, the approximate shift-invariance $\psi_2 \approx \psi_1(\cdot - 1/2)$ is clearly shown in Fig. 4.

(iii) For the cubic cardinal B-spline N_4, we make use of the VMR function

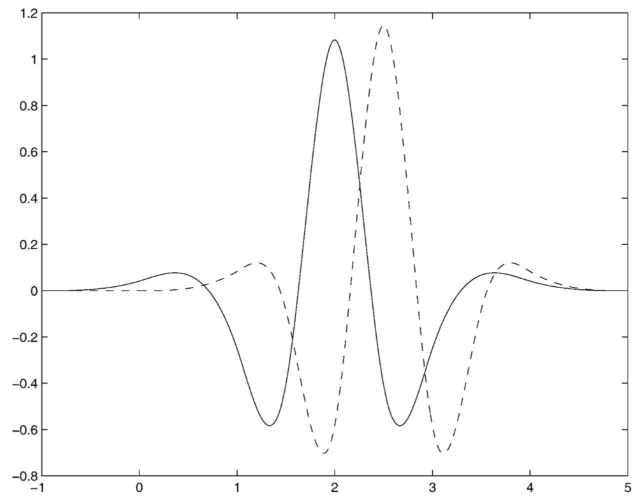
$$S(z) = 1 + \frac{1}{3}\left(2 - z - z^{-1}\right) + \frac{31}{360}\left(2 - z - z^{-1}\right)^2 + \frac{311}{15120}\left(2 - z - z^{-1}\right)^3$$

in order to reveal four vanishing moments. The factorization $\mathcal{M}(z) = \mathcal{R}(z^{-1})\mathcal{C}(z)\mathcal{R}^{\mathrm{T}}(z)$ can be formulated with

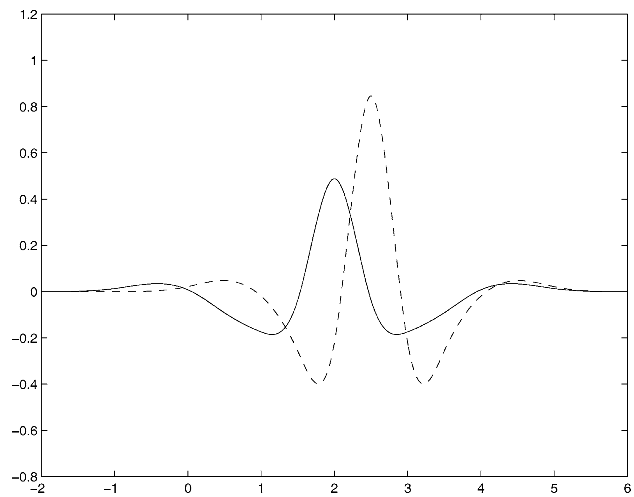
$$\mathcal{R}(z) = \begin{bmatrix} \left(\frac{1-z}{2}\right)^4 & 0 \\ 0 & \left(\frac{1+z}{2}\right)^4 \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \begin{bmatrix} 1 & \frac{1+z^2}{8} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2(1+z^{-2})}{5} & 1 \end{bmatrix} \quad \text{and}$$

$$\mathcal{C}(z) = \begin{bmatrix} \frac{27247+7775(z^2+z^{-2})}{945} & \frac{48346(1+z^{-2})}{4725} \\ \frac{48346(1+z^2)}{4725} & \frac{416856+2828(z^2+z^{-2})}{23625} \end{bmatrix}.$$

A pair of sibling frames $\{\psi_1, \psi_2\}$ and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ of symmetric functions is obtained by using a simple factorization of $\mathcal{C}$. This gives two-scale symbols



(a)



(b)

Fig. 5. Cubic spline sibling frame generators $\{\psi_1, \psi_2\}$ (a) and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ (b) with four vanishing moments.

$$
\begin{aligned}
Q_1(z) &= \frac{1}{4}\left(\frac{1-z}{2}\right)^4 \bigl(22 + 8(z + z^{-1}) + z^2 + z^{-2}\bigr),\\
Q_2(z) &= \tfrac{1}{2}\left(\frac{1-z}{2}\right)^4 \bigl(1 + 8z + z^2\bigr),\\
\widetilde{Q}_1(z) &= \frac{1}{18900}\left(\frac{1-z}{2}\right)^4 \bigl(132666 + 94712(z + z^{-1}) + 44494(z^2 + z^{-2})\\
&\quad + 12440(z^3 + z^{-3}) + 1555(z^4 + z^{-4})\bigr),\\
\widetilde{Q}_2(z) &= \frac{1}{9450}\left(\frac{1-z}{2}\right)^4 \bigl(61024z + 33045(1 + z^2) + 9952(z^{-1} + z^3)\\
&\quad + 1244(z^{-2} + z^4)\bigr).
\end{aligned}
$$

The sibling frame constructed here is shown in Fig. 5. A tight frame construction is considered in Section 5 where a new method for matrix factorization is presented. □

4.2. Inter-orthogonality

In addition to the maximum number of vanishing moments, we can require sibling frames to satisfy certain orthogonality relations.

Definition 2. The family $\Psi = \{\psi_1, \dots, \psi_n\} \subset L^2$ is *inter-orthogonal* if $W^i \perp W^j$, $i \neq j$, where $W^i = \operatorname{clos}_{L^2} \operatorname{span}\{\psi_i(x-k)\colon k \in \mathbb{Z}\}$.

We will study this property for generators ψ_i of a sibling frame. Standard computations using the Fourier transform of ψ_i show that $W^i \perp W^j$ is equivalent to

$$Q_i(z)\overline{Q_j(z)}B_\phi(z) + Q_i(-z)\overline{Q_j(-z)}B_\phi(-z) = 0, \quad |z| = 1. \tag{4.12}$$

We first show that inter-orthogonality requires that the number of generators be $n = 2$.

Proposition 2. *If $\psi_i \in V_1$, $1 \leqslant i \leqslant n$, are non-trivial and inter-orthogonal, then $n = 2$.*

Proof. Eq. (4.12) can be written in matrix form as

$$\begin{bmatrix} Q_1(z) & \dots & Q_n(z) \\ Q_1(-z) & \dots & Q_n(-z) \end{bmatrix}^* \begin{bmatrix} B_\phi(z) & 0 \\ 0 & B_\phi(-z) \end{bmatrix} \begin{bmatrix} Q_1(z) & \dots & Q_n(z) \\ Q_1(-z) & \dots & Q_n(-z) \end{bmatrix}$$
$$= \operatorname{diag}\bigl(\bigl(\left|Q_i(z)\right|^2 B_\phi(z) + \left|Q_i(-z)\right|^2 B_\phi(-z)\bigr)\colon 1 \leqslant i \leqslant n\bigr).$$

The matrix on the right has full rank n for some $z \in \mathbb{T}$, while the matrix on the left has rank at most 2. □

The existence of inter-orthogonal sibling frames with two generators (where inter-orthogonality is valid for one family) is assured by the next result.

Theorem 3. *For any compactly supported refinable function ϕ that satisfies* (2.3a)–(2.3c), *there exists a pair of sibling frames (ψ_1, ψ_2) and $(\tilde{\psi}_1, \tilde{\psi}_2)$ such that all of the four functions have compact support and the maximum number m of vanishing moments, and that (ψ_1, ψ_2) is inter-orthogonal.*

For the proof we make use of the following result in [24, Theorem 1.2].

Lemma 2. *Let u_1 and u_2 be Laurent polynomials that are nonnegative on $\mathbb{T}$ and have no common zeros in $\mathbb{C} \setminus \{0\}$. There exist Laurent polynomials v_1 and v_2 which are also nonnegative on $\mathbb{T}$, such that*

$$u_1(z)v_1(z) + u_2(z)v_2(z) = 1, \quad \text{for all } z \in \mathbb{C} \setminus \{0\}. \tag{4.13}$$

We also need the following lemma whose proof will be given later.

Lemma 3. *Let E be a Laurent polynomial with real coefficients and $E \geqslant 0$ on $\mathbb{T}$. Then E can be decomposed into Laurent polynomials with real coefficients,*

$$E(z) = D\bigl(z^2\bigr)E_0(z), \tag{4.14}$$

such that $E_0 \geqslant 0$ on $\mathbb{T}$, and that $E_0(z)$ and $E_0(-z)$ have no common zeros.

Proof of Theorem 3. Let S be a VMR Laurent polynomial with real coefficients and real-valued on $\mathbb{T}$, as in Theorem 2, such that the matrix $\mathcal{M}$ has the factorization

$$\mathcal{M}(z) = D_m(1/z)\mathcal{M}_0(z)D_m(z).$$

The objective is to find a suitable factorization

$$\mathcal{M}_0(z) = \begin{bmatrix} \overline{\tilde{q}_1(z)} & \overline{\tilde{q}_2(z)} \\ \tilde{q}_1(-z) & \tilde{q}_2(-z) \end{bmatrix} \begin{bmatrix} q_1(z) & q_1(-z) \\ q_2(z) & q_2(-z) \end{bmatrix} \tag{4.15}$$

so that the Laurent polynomials $Q_i(z) = ((1-z)/2)^m q_i(z)$ satisfy Eq. (4.12).

Assuming real coefficients for all Laurent polynomials, (4.12) can be expressed as

$$\left(\frac{1-z}{2}\right)^m\left(\frac{1-1/z}{2}\right)^m q_1(z)q_2(1/z)B_\phi(z) \\ +\left(\frac{1+z}{2}\right)^m\left(\frac{1+1/z}{2}\right)^m q_1(-z)q_2(-1/z)B_\phi(-z)=0, \quad z\neq 0. \tag{4.16}$$

Next we will show that there are solutions q_1 and q_2 of this equation so that

$$\det\begin{bmatrix} q_1(z) & q_1(-z) \\ q_2(z) & q_2(-z)\end{bmatrix} = -z. \tag{4.17}$$

For this purpose, we use the fact that the Laurent polynomial

$$E(z) := \left(\frac{1-z}{2}\right)^m\left(\frac{1-1/z}{2}\right)^m B_\phi(z)$$

in (4.16) has real coefficients and is non-negative on $\mathbb{T}$. By Lemma 3 we find a factorization

$$E(z) = d\big(z^2\big)E_0(z),$$

where E_0 has the same properties as E, and, in addition, $E_0(z)$ and $E_0(-z)$ have no common roots in $\mathbb{C}\setminus\{0\}$. The orthogonality relation (4.16) is automatically satisfied if we choose

$$q_1(z) = q_0(z)E_1(-z) \quad \text{and} \quad q_2(z) = zq_0(-1/z)E_2(-z),$$

where q_0 is an arbitrary Laurent polynomial with real coefficients and $E_1(z)E_2(z) = E_0(z)$. The factors E_1 and E_2 can be chosen to be non-negative on $\mathbb{T}$ and that none of the four functions $E_i(z)$, $E_i(-z)$, $1\leqslant i\leqslant 2$, have any common roots. Eq. (4.17) expressed for this choice of q_1, q_2 is equivalent to

$$\big|q_0(z)\big|^2E_1(z)E_2(-z) + \big|q_0(-z)\big|^2E_1(-z)E_2(z) = 1, \quad z\in\mathbb{T}.$$

Lemma 2 allows us to find a Laurent polynomial $r = |q_0|^2$ which satisfies this equation, and the Riesz Lemma gives a solution q_0.

The Laurent polynomials q_1, q_2 constructed so far define the family $\Psi = \{\psi_1, \psi_2\}$ which is inter-orthogonal, due to (4.16). Eq. (4.17) implies that

$$\begin{bmatrix} \overline{\tilde{q}_1(z)} & \overline{\tilde{q}_2(z)} \\ \tilde{q}_1(-z) & \tilde{q}_2(-z)\end{bmatrix} = \mathcal{M}_0(z)\begin{bmatrix} q_1(z) & q_1(-z) \\ q_2(z) & q_2(-z)\end{bmatrix}^{-1}$$

defines Laurent polynomials $\tilde{q}_1$, $\tilde{q}_2$ so that the factorization (4.15) of $\mathcal{M}_0$ is valid. Hence, we have found a sibling frame with m vanishing moments where Ψ is inter-orthogonal. □

We now give the proof of Lemma 3.

Proof of Lemma 3. Since E is a Laurent polynomial with real coefficients and is real on $\mathbb{T}$, it can be written as an algebraic polynomial e of the real variable $u = z + z^{-1} \in [-1, 1]$ with real coefficients. By assumption e is non-negative on $[-1, 1]$. Hence, we can find an integer $k \geqslant 0$ and an algebraic polynomial e_0 such that

$$e(u) = u^{2k}e_0(u), \quad e_0(0) > 0.$$

Obviously, e_0 is non-negative on $[-1, 1]$ as well.

Let $d_0(u)$ denote the greatest common divisor of $e_0(u)$ and $e_0(-u)$, which is normalized such that $d_0(0) = 1$. Since e_0 is non-negative on $[-1, 1]$, all roots of d_0 in $(-1, 1)$ must have even multiplicity. Therefore, d_0 is also non-negative on $[-1, 1]$. Moreover, it is easy to see that $d_0(-u)$ is a common divisor of $e_0(u)$ and $e_0(-u)$ as well. Hence, $d_0(u)$ must be a constant multiple of $d_0(-u)$, and the positivity at 0 gives $d_0(u) = d_0(-u)$. This implies that d_0 is an algebraic polynomial in even powers of u. In other words, we obtain a factorization

$$E(z) = e(u) = u^{2k}d_1\big(u^2\big)e_1(u),$$

with algebraic polynomials d_1 and e_1 that are non-negative on $[-1,1]$, such that $e_1(u)$ and $e_1(-u)$ have no common zeros in $\mathbb{C}$. The factorization in the lemma is then obtained by defining $D(z^2) := u^{2k} d_1(u^2)$ and $E_0(z) := e_1(u)$. By construction, these Laurent polynomials have real coefficients and E_0 is non-negative on $\mathbb{T}$. Due to the algebraic relation $E_0(-z) = e_1(-u)$, the Laurent polynomials $E_0(z)$ and $E_0(-z)$ have no common zeros. □

Example 2. For the linear cardinal B-spline N_2 with integer knots, we use the same VMR Laurent polynomial $S(z) = 1 + (2 - z - z^{-1})/6$ as in the example in Section 4.1. The two-scale symbols of the inter-orthogonal frame generators ψ_1 and ψ_2 are formulated as $Q_i(z) = ((1-z)/2)^2 q_i(z)$, $i = 1, 2$, where

$$q_1(z) = \left(\frac{2+z+z^{-1}}{4}\right)^2 q_0(z), \qquad q_2(z) = \frac{z(4-z-z^{-1})}{6} q_0(-1/z).$$

The polynomial q_0 has the form $q_0(z) = az^2 + bz + c$ with coefficients

$$a = 1/4 + \frac{1}{12}\sqrt{57} - \frac{1}{12}\sqrt{42 + 6\sqrt{57}} \approx 0.1005,$$
$$c = 1/4 + \frac{1}{12}\sqrt{57} + \frac{1}{12}\sqrt{42 + 6\sqrt{57}} \approx 1.6578,$$
$$b = 1/2 - \frac{1}{6}\sqrt{57} \approx -0.7583.$$

The two-scale symbols for the dual pair are obtained in the form of $\widetilde{Q}_i(z) = ((1 - z)/2)^2 \tilde{q}_i(z)$, $i = 1, 2$, where

$$\tilde{q}_1(z) := -zS\big(z^2\big) q_2(1/z) - zA(z) q_2(-1/z),$$
$$\tilde{q}_2(z) := zA(z) q_1(-1/z) + zS(z^2) q_1(1/z),$$

and $A(z) = (24 + 8(z + z^{-1}) + z^2 + z^{-2})/24$ is the first diagonal entry of the reduced matrix $\mathcal{M}_0$. Fig. 6 depicts the graphs of the generators $\{\psi_1, \psi_2\}$ (a) and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ (b). □

5. Tight frames with two generators

In this section we show that tight affine frames with two compactly supported generators $\psi_1, \psi_2 \in V_1$ exist for any refinable function ϕ whose integer shifts are stable, such that both generators have the maximal order m of vanishing moments, where m is the order of the zero $z = -1$ of the two-scale polynomial P. We include the detailed description of a constructive procedure for the tight frame generators ψ_1 and ψ_2.

One part of this procedure consists of extending the spectral factorization of trigonometric polynomials, as described in [28, pp. 117–118], to matrix-valued Laurent polynomials

$$\mathcal{M}(z) = \sum_{k=-N}^{N} A_k z^k$$

that are positive semidefinite on $\mathbb{T}$ and whose coefficients A_k are 2×2 matrices with real entries. The underlying theoretical result is a well-known generalization of the Fejér–Riesz Theorem which was obtained by Rosenblatt [32]. The following version of the result together with a generalization to operator-valued polynomials as well as several useful historical remarks can be found in the monograph [33, Section 6.6].

Theorem C. *Let $\mathcal{M}(z) = \sum_{k=-N}^{N} A_k z^k$ be a trigonometric polynomial with coefficients $A_k \in \mathbb{C}^{n\times n}$ such that $\mathcal{M}$ is positive semidefinite on $\mathbb{T}$. Then there exists an outer function $\mathcal{R}(z) = \sum_{k=0}^{N} B_k z^k$ with coefficients $B_k \in \mathbb{C}^{n\times n}$, such that*

$$\mathcal{M}(z) = \mathcal{R}^*(z)\mathcal{R}(z), \quad z \in \mathbb{T}. \tag{5.1}$$

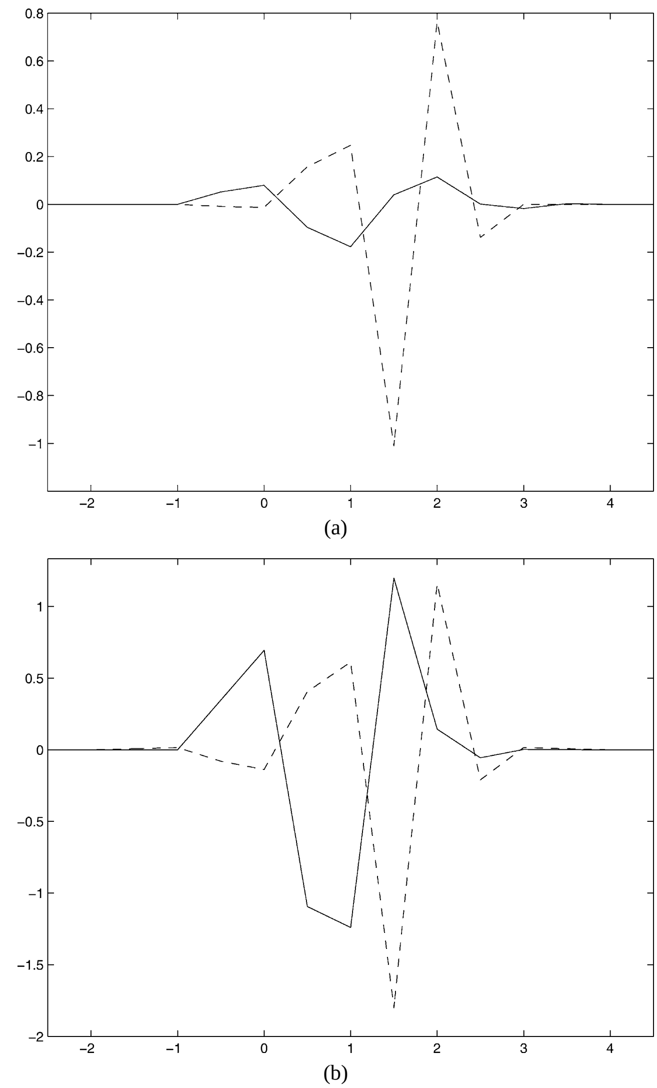


Fig. 6. Linear spline sibling frames with two vanishing moments; interorthogonal generators $\{\psi_1, \psi_2\}$ (a), dual generators (b).

The notion of inner and outer operator-valued functions is explained in [33]. Several numerical procedures for the construction of the factorization (5.1) are described in [8,25]. Some of these methods employ an equivalent representation of the matrix polynomial $\mathcal{M}$ as a biinfinite block Toeplitz matrix and use a Wiener–Hopf type method computing the Cholesky factors of finite compressions of this matrix, see [25]. Another method described in [25] uses a relatively complex spectral factorization technique in order to obtain an LDU-decomposition of $\mathcal{M}$. Its simplification for the case of symmetry, definiteness, and low dimension of the matrix polynomial are not obvious to us. For this reason we include a simpler construction of a spectral factorization (5.1) where $\mathcal{M}$ is a 2×2 matrix polynomial that is positive semidefinite on $\mathbb{T}$. Our construction requires only the spectral factorization of univariate trigonometric polynomials and linear algebra techniques.

Our construction is based on a reduced form of the matrix polynomial that is obtained from the following lemma.

Lemma 4. *Let*

$$\mathcal{M}(z) = \begin{bmatrix} A(z) & B(z) \\ B(1/z) & C(z) \end{bmatrix}$$

be a matrix of Laurent polynomials with real coefficients. If $\mathcal{M}$ is positive semidefinite on $\mathbb{T}$, then there exists a Laurent polynomial d with real coefficients, such that

$$\mathcal{M}(z)=\begin{bmatrix} d(z) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix} A_0(z) & B_0(z)\\ B_0(1/z) & C(z)\end{bmatrix}\begin{bmatrix} d(1/z) & 0\\ 0 & 1\end{bmatrix},$$

where A_0 and B_0 are Laurent polynomials with no common roots in $\mathbb{C}\setminus\{0\}$. Moreover, the matrix in the middle of the above equation is positive semidefinite on $\mathbb{T}$ and, in particular, $A_0(z)=\sum_{k=0}^{N} a_k(z+z^{-1})^k$ is strictly positive on $\mathbb{T}$.

Proof. If A and B have no common roots in $\mathbb{C}\setminus\{0\}$, the factorization in the lemma is valid for $d\equiv 1$. Otherwise, let us denote by $\mathcal{Z}$ the collection of all common roots of A and B, counting each root with the minimum of both multiplicities as a root of A and B, respectively. Note that both A and B have real coefficients. Furthermore, A must be non-negative on $\mathbb{T}$ by virtue of our assumptions on $\mathcal{M}$, and that $A(z)=A(1/z)$, $|z|=1$, is satisfied.

For any $w\in\mathcal{Z}$, we will find a factor $\tilde d_w$ in each of the following three cases such that

$$A(z)=\tilde d_w(z)\tilde d_w(1/z)A_1(z),\qquad B(z)=\tilde d_w(z)B_1(z). \tag{5.2}$$

Here $\tilde d_w$, A_1, and B_1 are Laurent polynomials with real coefficients and $\tilde d_w(w)=0$. By construction, A_1 is non-negative on $\mathbb{T}$. By proceeding in this manner all common zeros of A and B can be eliminated:

(a) If w is not real and $|w|\neq 1$, then $\tilde d_w(z)=(z-w)(z-\bar w)$ is a Laurent polynomial with real coefficients which divides both A and B. Moreover, $\tilde d_w(1/z)$ is a factor of A having zeros $1/w$ and $1/\bar w$ which are distinct from w, $\bar w$. This gives (5.2).
(b) If $w\in\mathbb{R}\setminus\{-1,1\}$ we have $A(w)=A(1/w)=0$. This gives (5.2), where $\tilde d_w(z)=(z-w)$.
(c) If $w\in\mathbb{T}$, the multiplicity k_A of the root w of A is even, since A is non-negative on $\mathbb{T}$. If $w\notin\{-1,1\}$, we let

$$\tilde d_w(z)=z^{-1}(z-w)(z-\bar w)=\bigl(z+1/z-(w+\bar w)\bigr).$$

Obviously, $\tilde d_w$ has real coefficients, and $\tilde d_w^2(z)=\tilde d_w(z)\tilde d_w(1/z)$ is a factor of A. This gives (5.2). For the remaining case $w\in\{-1,1\}$, we make use of $w=1/w$ in the formulation

$$A(z)=(1-wz)\bigl(1-wz^{-1}\bigr)A_1(z),\qquad B(z)=(1-wz)B_1(z).$$

Again, (5.2) is established for this case.

After applying this procedure finitely many times we obtain a factorization

$$A(z)=d(z)d(1/z)A_0(z),\qquad B(z)=d(z)B_0(z),$$

where all Laurent polynomials have real coefficients, and A_0 and B_0 have no common roots in $\mathbb{C}\setminus\{0\}$. Obviously,

$$B(1/z)=d(1/z)B_0(1/z)$$

is also valid. The last two equations give the factorization in Lemma 4. It is also obvious that the matrix in the middle of this factorization is positive semidefinite. Hence, its diagonal entry A_0 is non-negative on $\mathbb{T}$, and this implies that it is an algebraic polynomial in $u:=z+z^{-1}$. Moreover, if $w\in\mathbb{T}$ were a root of A_0, the definiteness of the matrix would imply that $B_0(w)B_0(1/w)=0$. This would give a common root (w or $\bar w$) of A_0 and B_0 which does not exist. This completes the proof of the lemma. □

The next theorem gives a new construction based on univariate spectral factorization for the matrix decomposition (5.1). Moreover, we establish a one-to-one correspondence between all factorizations of the form (5.1) whose polynomial degree is restricted with the set of all solutions of a *linear* homogeneous system of equations (5.4)–(5.5) and a simple quadratic side condition (5.6). Therefore, the matrix factorization (5.1) can be determined using methods of linear algebra.

We define the degree of a Laurent polynomial $\sum_{k=N_1}^{N_2} c_k z^k$, with $N_1 \leqslant N_2$ and $c_{N_k} \neq 0$ for $k = 1, 2$, to be $N_2 - N_1$.

Theorem 4. *Let*

$$\mathcal{M}(z) = \begin{bmatrix} A(z) & B(z) \\ B(1/z) & C(z) \end{bmatrix}$$

be a matrix of Laurent polynomials with real coefficients which is positive semidefinite on $\mathbb{T}$, and suppose that $A(z) = \sum_{k=0}^{N} a_k (z + z^{-1})^k$ and B have no common roots in $\mathbb{C} \setminus \{0\}$, and $a_N \neq 0$. Then there exist four Laurent polynomials u_1, u_2, v_1, v_2 with real coefficients, with u_1 and u_2 of degree at most N, such that

$$\mathcal{M}(z) = \begin{bmatrix} u_1(1/z) & u_2(1/z) \\ v_1(1/z) & v_2(1/z) \end{bmatrix} \begin{bmatrix} u_1(z) & v_1(z) \\ u_2(z) & v_2(z) \end{bmatrix} =: \mathcal{R}^{\mathrm{T}}(1/z)\mathcal{R}(z). \tag{5.3}$$

The quadruple (u_1, u_2, v_1, v_2) is a solution of the linear homogeneous system

$$B(z)u_1(z) - d(z)u_2(1/z) - A(z)v_1(z) = 0, \tag{5.4}$$

$$d(1/z)u_1(z) + B(1/z)u_2(1/z) - A(z)v_2(1/z) = 0 \tag{5.5}$$

and

$$u_1^2(1) + u_2^2(1) = A(1), \tag{5.6}$$

where d is a Laurent polynomial such that

$$d(z)d(1/z) = \det \mathcal{M}(z). \tag{5.7}$$

Conversely, any Laurent polynomial solution (u_1, u_2, v_1, v_2) of (5.4)–(5.6)*, with u_1 and u_2 of degree at most N and d as in* (5.7)*, defines a factorization* (5.3) *of $\mathcal{M}$.*

Proof. The existence of algebraic polynomials (u_1, u_2, v_1, v_2) that define a factorization (5.3) is part of the general result of Theorem C. The assertion of Theorem 4 is slightly stronger as far as the degree of the polynomials u_1 and u_2 is concerned. The proof is organized as follows. First, we show the equivalence of the matrix factorization (5.3) and the system of equations (5.4)–(5.6) under the assumption that u_1 and u_2 have degree at most N. Then, we prove existence of solutions (u_1, u_2, v_1, v_2) of (5.4)–(5.6) that meet the assumption on the degree of u_1 and u_2. We frequently use the fact that A has no zeros on $\mathbb{T}$, which follows from Lemma 4 and our assumptions on $\mathcal{M}$.

Let us assume that a factorization of $\mathcal{M}$ in (5.3), with

$$\mathcal{R} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix},$$

is defined where u_1 and u_2 have degree at most N. Then $d = \det \mathcal{R} = u_1 v_2 - u_2 v_1$ satisfies (5.7), and the equation

$$u_1(z)u_1(1/z) + u_2(z)u_2(1/z) = A(z)$$

implies (5.6). In order to prove (5.4)–(5.5), we let

$$\begin{aligned} \alpha(z) &:= B(z)u_1(z) - d(z)u_2(1/z) - A(z)v_1(z), \\ \beta(z) &:= d(z)u_1(1/z) + B(z)u_2(z) - A(z)v_2(z). \end{aligned} \tag{5.8}$$

It follows from (5.3) and (5.7) that

$$\alpha(z)u_1(1/z) + \beta(z)u_2(1/z) = 0, \qquad \alpha(z)v_1(1/z) + \beta(z)v_2(1/z) = 0. \tag{5.9}$$

This is a homogeneous system of linear equations for α and β, whose determinant $d(1/z)$ is non-zero for almost all $z \in \mathbb{T}$. Therefore, $\alpha = \beta = 0$ is the only Laurent polynomial solution, and (5.4)–(5.5) must be satisfied.

Conversely, let d be any Laurent polynomial with real coefficients that satisfies (5.7). Moreover, let Laurent polynomials (u_1, u_2, v_1, v_2) with real coefficients be given, with u_1 and u_2 of degree at most N, such that (5.4)–(5.6) is satisfied. Hence, the

Laurent polynomials α and β, as defined in (5.8), are zero. After reordering the terms $u_1(1/z)\alpha(z) + u_2(1/z)\beta(z)$ we obtain

$$B(z)\big[u_1(z)u_1(1/z) + u_2(z)u_2(1/z)\big] = A(z)\big[v_1(z)u_1(1/z) + v_2(z)u_2(1/z)\big]. \quad (5.10)$$

Since A and B have no common roots in $\mathbb{C} \setminus \{0\}$, by assumption, this shows that $[u_1(z)u_1(1/z) + u_2(z)u_2(1/z)]$ is divisible by $A(z)$; in other words

$$u_1(z)u_1(1/z) + u_2(z)u_2(1/z) = p(z)A(z), \quad (5.11)$$

for some Laurent polynomial p. By the assumption that the degree of u_1 and u_2 cannot exceed N, the left hand side of (5.11) is a Laurent polynomial of the form $\sum_{k=0}^{N} c_k(z + 1/z)^k$ with real coefficients c_k, $0 \leqslant k \leqslant N$. Consequently, p must be constant, and (5.6) implies that $p = 1$. By combining (5.10) and (5.11) we obtain

$$\begin{aligned} &u_1(z)u_1(1/z) + u_2(z)u_2(1/z) = A(z), \\ &v_1(z)u_1(1/z) + v_2(z)u_2(1/z) = B(z), \end{aligned} \quad (5.12)$$

which yields one part of the matrix factorization (5.3). If we operate analogously on α and β in (5.8) by taking the combination $v_2\alpha - v_1\beta$, we obtain

$$B(z)\big[u_1(z)v_2(z) - u_2(z)v_1(z)\big] = d(z)\big[v_1(z)u_1(1/z) + v_2(z)u_2(1/z)\big] = d(z)B(z),$$

where the last equation follows from the second relation in (5.12). Hence, we have

$$u_1(z)v_2(z) - u_2(z)v_1(z) = d(z). \quad (5.13)$$

Furthermore, the combination $v_1(1/z)\alpha + v_2(1/z)\beta$ gives

$$A(z)\big[v_1(z)v_1(1/z) + v_2(z)v_2(1/z)\big] = B(z)B(1/z) + d(z)d(1/z) = A(z)C(z).$$

Here we used the second relation in (5.12) and (5.13), with z replaced by $1/z$, together with the fact that d satisfies (5.7). Now, we can conclude that

$$v_1(z)v_1(1/z) + v_2(z)v_2(1/z) = C(z). \quad (5.14)$$

Eqs. (5.12) and (5.14) give the matrix factorization (5.3).

In the remaining part of the proof, we show that Laurent polynomials (u_1, u_2, v_1, v_2) with real coefficients exist, with u_1 and u_2 of degree at most N, which satisfy (5.4)–(5.6). We begin by constructing algebraic polynomials u_1 and u_2 such that the Laurent polynomial

$$B(z)u_1(z) - d(z)u_2(1/z) \quad (5.15)$$

is divisible by A. Note that $z^N A(z)$ is an algebraic polynomial of exact degree $2N$. All its roots lie in $\mathbb{C} \setminus \{0\}$. Let w be a root of A of multiplicity k. Then $(z - w)^k$ is a factor of the Laurent polynomial (5.15) if and only if

$$\frac{d^\nu}{dz^\nu}\big[B(z)u_1(z) - d(z)u_2(1/z)\big]_{z=w} = 0 \quad \text{for all } 0 \leqslant \nu \leqslant k-1. \quad (5.16)$$

If w is real, (5.16) specifies k real and homogeneous equations for the unknown coefficients of u_1 and u_2. If w is not real, the real and imaginary parts of (5.16) give $2k$ real and homogeneous equations for the unknown coefficients of u_1 and u_2 which are equivalent to the fact that $[(z - w)(z - \bar{w})]^k$ is a factor of the Laurent polynomial (5.15). The total number of equations in (5.16), taking into consideration all of the roots of A, is $2N$. Therefore, non-trivial algebraic polynomials u_1 and u_2 of degree at most N exist such that A divides the Laurent polynomial in (5.15); in other words, there exist Laurent polynomials u_1, u_2, v_1 with real coefficients, with u_1 and u_2 of degree at most N, such that

$$B(z)u_1(z) - d(z)u_2(1/z) - A(z)v_1(z) = 0.$$

The triple (u_1, u_2, v_1) defines a solution of Eq. (5.4).

Let us note here that any multiple of (u_1, u_2, v_1) provides a solution of (5.4) as well. Furthermore, any common roots of $u_1(z)$ and $u_2(1/z)$ which lie on $\mathbb{T}$ can be dropped, because A does not vanish on $\mathbb{T}$. Hence, we can find a normalized solution (u_1, u_2, v_1) of (5.4) which also satisfies (5.6).

Finally, we show that this choice of (u_1, u_2, v_1) leads to a solution (u_1, u_2, v_2) of Eq. (5.5). Indeed, multiplication by $d(1/z)$ on both sides of (5.4) and Eq. (5.7) gives

$$B(z)d(1/z)u_1(z) - \det\mathcal{M}(z)u_2(1/z) = A(z)d(1/z)v_1(z).$$

Thus, we obtain

$$B(z)\big[d(1/z)u_1(z) + B(1/z)u_2(1/z)\big] = A(z)\big[d(1/z)v_1(z) + C(z)u_2(1/z)\big]. \quad (5.17)$$

Now, by the assumption that A and B have no common roots, the factor inside the brackets on the left-hand side of (5.17) must be divisible by A. We can conclude that (5.4) implies (5.5), with a suitable choice of the Laurent polynomial v_2. This shows the existence of Laurent polynomials (u_1, u_2, v_1, v_2) that satisfy (5.4)–(5.6) and, by the equivalence that we proved before, the existence of the matrix factorization (5.3).

Thus we have completed the proof of Theorem 4. □

Remark 5. Construction of the factor $\mathcal{R}$ in (5.3) is based on knowledge of a factorization (5.7) of the positive Laurent polynomial $\det\mathcal{M}$. The additional steps can be carried out by using methods of elementary linear algebra. In this regard, the complexity of the method is comparable to the univariate spectral factorization technique that is based on the fundamental theorem of algebra, see [28]. In particular, the construction circumvents the use of Gröbner basis methods which, at a first glance, appear to be necessary to solve equations (5.12). No claim is made that the factor $\mathcal{R}$ in Theorem 4 is an outer function as in the abstract Theorem C. A more general construction for all matrix polynomials of any (finite) size is currently under investigation by the authors.

In the following, we demonstrate the effective procedure by revisiting Example 1 in Section 4.

Example 3. As in Section 4, the refinable function ϕ is chosen to be the cardinal B-spline N_m of order m with integer knots, and supported on the interval $[0, m]$. The vanishing-moment recovery function $S(z)$ in (4.11) exhibits a positive definite matrix $\mathcal{M}_0$ that satisfies the assumptions of Theorem 4. For $m = 2$ and $m = 3$, the matrix $\mathcal{M}_0$ can be reduced to a diagonal matrix by using the Euclidean algorithm described in Appendix A. Hence, a factorization (5.3) can even be found without appealing to the constructive method of Theorem 4. For $m = 4$, however, the reduction by the Euclidean algorithm in Appendix A leaves a non-diagonal matrix

$$\mathcal{C}(z) = \begin{bmatrix} \frac{27247+7775(z^2+z^{-2})}{945} & \frac{48346(1+z^{-2})}{4725} \\ \frac{48346(1+z^2)}{4725} & \frac{416856+2828(z^2+z^{-2})}{23625} \end{bmatrix}.$$

Instead of defining a pair of symmetric sibling frames as in Section 4, the method of Theorem 4 can be employed for the construction of non-symmetric tight frame generators (ψ_1, ψ_2). If we substitute x for z^2 in $\mathcal{C}(z)$, the parameter N in Theorem 4 is 1. Solutions (u_1, u_2, v_1) of Eq. (5.4), which are algebraic polynomials of degree at most 1, can be chosen to have the form

$$\begin{aligned} u_1(x) &= c\big[7775\big(d_2 - d_1(1+x) + d_0 x\big) + 19472(d_0 + d_2 x)\big] \\ &= 3.15315x + 2.60930, \\ u_2(x) &= c[26928722x/225] = 3.47592x, \\ v_1(x) &= c\big[48346(d0 - d1 + d2)/5\big] = 4.12182. \end{aligned}$$

Their coefficient sequences (a total of 6 unknowns for u_1, u_2, and v_1) are chosen from the null space of a system of four linear equations. Here, the coefficients $(d_0, d_1, d_2) = (-2.07544, -17.2278, -0.474532)$ stem from the univariate factorization

$$\det\mathcal{C}(z) = \big(d_0 + d_1 x + d_2 x^2\big)\big(d_0 + d_1 x^{-1} + d_2 x^{-2}\big)$$

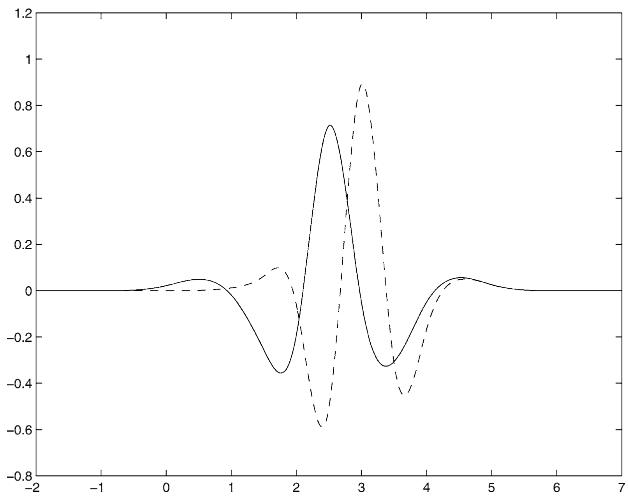


Fig. 7. Cubic spline tight frame generators ψ_1 (solid line) and ψ_2 (dashed line) with four vanishing moments.

and can be computed using a separate procedure. The constant $c = 2.90427 \times 10^{-5}$ is used to guarantee condition (5.6). Finally, the Laurent polynomial $v_2(x) = -0.150495x - 0.795400$ is computed using the relation

$$B(x) - u_1(1/x)v_1(x) = u_2(1/x)v_2(x).$$

This gives the factorization (5.3) of $\mathcal{C}$. If we combine this new factorization with the previous steps (factorization of moments, Euclidean algorithm) that were performed in Section 4, the two-scale Laurent polynomials of non-symmetric tight frame generators (ψ_1, ψ_2) become

$$Q_i(z) = \left(\frac{1-z}{2}\right)^4 q_i(z), \quad i = 1, 2,$$

and the coefficient sequences of q_i are as follows:

k	-2	-1	0	1	2	3	4
q_1	0.130465	1.04372	3.54312	6.42680	4.11416	1.26126	0.157657
q_2			0.074371	0.594967	3.70527	1.23987	0.154984

This example of minimally supported tight frame generators (ψ_1, ψ_2) was first considered in [14]. Graphs of ψ_1 and ψ_2 are depicted in Fig. 7. □

Remark 6. We like to point out that our linear algebra approach was already described in the first draft of the manuscript. In fact, the manuscript submitted to ACHA for publication contained only minor modifications of the draft distributed to others. The only significant change occurs in the above example, where the degrees of the polynomials q_1 and q_2 were reduced from 13, 11 to 11, 9 after we had a chance to see the manuscript [14]. We thank the authors of [14] for providing us their manuscript before it was submitted for publication.

Remark 7. Example 3 demonstrates a general procedure that Theorem 4 makes available. It explains how the spectral factorization of $\det \mathcal{M}$ and the solution of the linear system (5.4) can be decoupled. The actual coefficients of the Laurent polynomial $d(z)$ in (5.4) are only needed for the normalization in (5.6).

In order to use the result of Theorem 4 for our construction of tight frames, we first need to find a positive semidefinite matrix

$$\mathcal{M}(z) := \begin{bmatrix} S(z) - S(z^2)P(z)P(1/z) & -S(z^2)P(1/z)P(-z) \\ -S(z^2)P(z)P(-1/z) & S(-z) - S(z^2)P(-z)P(-1/z) \end{bmatrix},$$

as in (4.2), by a suitable choice of the VMR function S. Note that the matrix $\mathcal{M}$ is positive semidefinite on $\mathbb{T}$ if and only if

$$S(z) - S(z^2)\left|P(z)\right|^2 \geqslant 0 \tag{5.18}$$

and

$$\Delta\left(z^2\right) := S(z)S(-z) - S\left(z^2\right)\left[S(-z)\left|P(z)\right|^2 + S(z)\left|P(-z)\right|^2\right] \geqslant 0. \tag{5.19}$$

However, for nonnegative S, the condition in (5.19) already implies (5.18). Therefore, it is sufficient to find a Laurent polynomial S that is nonnegative on $\mathbb{T}$ and satisfies (5.19), in order to construct a positive semidefinite matrix $\mathcal{M}$ in (4.2). By rewriting (5.19) as

$$\Delta\left(z^2\right) = S(z)S(-z)S\left(z^2\right)\left[\frac{1}{S(z^2)} - \frac{|P(z)|^2}{S(z)} - \frac{|P(-z)|^2}{S(-z)}\right],$$

we see that for $S \geqslant 0$, the positivity condition in (5.19) is equivalent to the positivity condition

$$\frac{1}{S(z^2)} - \frac{|P(z)|^2}{S(z)} - \frac{|P(-z)|^2}{S(-z)} \geqslant 0,$$

which is linear in $1/S$.

Corollary 1. *Let ϕ be a compactly supported refinable function that satisfies* (2.3a)–(2.3c), *and S a Laurent polynomial with real coefficients that satisfies $S(1) = 1$ and $S(z) > 0$ for all $z \in \mathbb{T}$. Then* (5.19) *is a necessary and sufficient condition for S to be a VMR function.*

Remark 8. A different positivity condition for S is established in [14] for the existence of compactly supported wavelet tight frames associated with ϕ, namely

$$A(z) := S(z) - S\left(z^2\right)\left(\left|P(z)\right|^2 + \left|P(-z)\right|^2\right) \geqslant 0. \tag{5.20}$$

We remark that this (linear) condition (in S) is a sufficient but not a necessary condition. In fact, there is a large class of compactly supported refinable functions with two-scale symbols $P(z)$ that allow the construction of tight frames with compactly supported frame generators, for which there are no Laurent polynomials $S(z)$ that satisfy (5.20), with $S(1) = 1$ and $S(z) \geqslant 0, z \in \mathbb{T}$. As a clarification of this point, we include the following example.

Example 4. Let ϕ be a refinable function with two-scale symbol $P(z)$ that satisfies

$$\left|P(z)\right|^2 + \left|P(-z)\right|^2 \geqslant 1, \quad z \in \mathbb{T}, \tag{5.21}$$

and not identically equal to one on the unit circle $\mathbb{T}$. Examples of such refinable functions include those provided by the dual scaling functions $\phi_{m,n}$ which are biorthogonal to the cardinal B-spline N_m of order $m \geqslant 2$ and have n vanishing moments, $1 \leqslant n \leqslant m$. Indeed, if P denotes the two-scale symbol of $\phi_{m,n}$, then

$$P(z)\widetilde{P}(1/z) + P(-z)\widetilde{P}(-1/z) \equiv 1, \quad z \in \mathbb{T},$$

where $\widetilde{P} := (1 + z/2)^m$, so that

$$\left|P(z)\right|^2 + \left|P(-z)\right|^2 > 1, \quad z \in \mathbb{T} \setminus \{1, -1\},$$

since $|\widetilde{P}(z)|^2 + |\widetilde{P}(-z)|^2 = \cos^{2m}(\omega/2) + \sin^{2m}(\omega/2) < 1$, $z \in \mathbb{T} \setminus \{1, -1\}$.

We claim that under condition (5.21), condition (5.20) can never be satisfied for any Laurent polynomial S satisfying $S(1) = 1$ and $S(z) > 0$ on $\mathbb{T}$. This statement can be justified in two steps, as follows:

(i) First we show that (5.20) and (5.21) imply that $S \geqslant 1$ on the unit circle. To see this, we note, by continuity, that it is sufficient to verify that $S(z) \geqslant 1$ for all $z \in \mathbb{T}$ for which there is an $n \in \mathbb{N} \cup \{0\}$ such that $z^{2^n} = 1$. We prove this by induction on n. For $n = 0$, the condition $S(1) = 1$ gives the desired inequality. Assume that $S(z) \geqslant 1$ for all z such that $z^{2^n} = 1$. Then, for any w with $w^{2^{n+1}} = 1$, we have, by applying (5.20) and (5.21),

$$S(w) \geqslant S(w^2) \underbrace{\left(\left|P(w)\right|^2 + \left|P(-w)\right|^2\right)}_{\geqslant 1} \geqslant S(w^2) \geqslant 1.$$

This shows that the inequality $S(z) \geqslant 1$ holds for all "dyadic" roots of unity. By continuity of S, we find $S(z) \geqslant 1$ on $\mathbb{T}$.

(ii) Let z be a dyadic root of unity such that $|P(z)|^2 + |P(-z)|^2 > 1$. Clearly, z cannot be 1. We choose a sequence $(w_n)_{n\geqslant 1}$ such that $w_n^2 = w_{n-1}, \ldots, w_1^2 = z =: w_0$, and $\lim_{n\to\infty} w_n = 1$. The positivity of A and $S \geqslant 1$ imply that

$$S(w_n) \geqslant \prod_{k=0}^{n} \left(\left|P(w_k)\right|^2 + \left|P(-w_k)\right|^2\right) S(z^2) \geqslant \left|P(z)\right|^2 + \left|P(-z)\right|^2 =: c_0 > 1,$$

and, therefore, the sequence $\{S(w_n)\}, n = 1, 2, \ldots,$ which is bounded below by c_0, cannot converge to 1.

This shows that the conditions (5.20), $S(1) = 1$, and the continuity of S cannot hold simultaneously. In other words, no Laurent polynomial S, which is non-negative on $\mathbb{T}$, exists, such that $A(z)$ in (5.20) is non-negative for $z \in \mathbb{T}$. □

However, there does exist some Laurent polynomial S with $S(1) = 1$ and $S(z) > 0$ on $\mathbb{T}$, such that condition (5.19) holds for $\phi_{m,n}$ according to the following theorem.

One possible way for finding such an S is described in the following.

Theorem 5. *Let ϕ be a compactly supported refinable function that satisfies* (2.3a)–(2.3c). *If the Laurent polynomials $P(z)$ and $P(-z)$, with P in* (2.2), *have no common roots and P satisfies Cohen's condition, then there is a Laurent polynomial S with real coefficients that satisfies $S(1) = 1$, $S(z) > 0$ for all $z \in \mathbb{T}$, and Eqs.* (4.4) *and* (5.19). *In other words, S is a VMR Laurent polynomial function such that the matrix $\mathcal{M}$ in* (4.2) *is positive semidefinite.*

The construction of S is based on properties of the transfer operator

$$T_{|P|^2}(f)(z^2) = \left|P(z)\right|^2 f(z) + \left|P(-z)\right|^2 f(-z), \tag{5.22}$$

which is a positive operator acting on certain finite-dimensional subspaces of Laurent polynomials. (Here, we restrict the variable z to $\mathbb{T}$; hence, all Laurent polynomials can be identified with trigonometric polynomials.) The transfer operator was analyzed in connection with the study of smoothness and stability properties of refinable functions (see [22,23]). It is easy to see that

$$E_N := \left\{ \sum_{j=0}^{N} f_j \left(z^j + z^{-j}\right):\ f_j \in \mathbb{R} \right\}$$

is an invariant subspace of $T_{|P|^2}$, where $N = N_2 - N_1$ refers to the degree of the symmetric Laurent polynomial $|P|^2$ (see (2.2)). Moreover, the subspaces

$$E_{N,k} := \left\{ f \in E_N:\ f(z) = \mathcal{O}\left(|z-1|^k\right) \text{ near } z = 1 \right\}, \quad 1 \leqslant k \leqslant 2m, \tag{5.23}$$

are invariant subspaces.

The notion of positive cones naturally restricts to the spaces $E_{N,2k}$, $0 \leqslant k \leqslant m$, with topology defined by the norm

$$\|f\|_{2k} := \max_{z\in\mathbb{T}} \left| f(z)(1-z)^{-2k} \right|, \quad f \in E_{N,2k}.$$

The cone of non-negative functions in $E_{N,2k}$, denoted by $P_{N,2k} := \{f \in E_{N,2k}:\ f \geqslant 0 \text{ on } \mathbb{T}\}$, is closed, convex, and generates $E_{N,2k}$ in the usual sense that $P_{N,2k} - P_{N,2k}$ is the full space. Its interior consists of all functions $f(z) \in E_{N,2k}$ that are strictly positive on $\mathbb{T} \setminus \{1\}$ and have a zero of exact order $2k$ at 1. A well drafted extension of notions of irreducibility and Perron–Frobenius theory of positive matrices in [16] to positive linear operators on finite-dimensional vector spaces can be found in [27,35].

The aforementioned notions are essential in order to discuss the existence of positive eigenfunctions of the transfer operator $T = T_{|P|^2}$ acting on $E_{N,2k}$, where positivity $f > 0$ means that f is an interior point of the positive cone $P_{N,2k}$. Let us first analyze the irreducibility of the transfer operator. According to [27], irreducibility is defined as the following property: if $Tf \leqslant \alpha f$ for some positive number α and some $f \geqslant 0$, $f \not\equiv 0$, then $f > 0$. We need the following.

Lemma 5. *The operator $T_{|P|^2}$, restricted to its invariant subspace $E_{N,2k}$, is irreducible with respect to the cone $P_{N,2k}$ of positivity, if and only if $P(z)$ and $P(-z)$ have no common roots on $\mathbb{T}$ and P satisfies Cohen's condition.*

Proof. Let us assume that $P(z)$ and $P(-z)$ have no common roots on $\mathbb{T}$ and that P satisfies Cohen's condition. Since the arguments are similar to those in the proof of Proposition 1, we only give a short outline here. Let $f \in E_{N,2k}$, $f \geqslant 0$ and $f \not\equiv 0$ be given, such that $T_{|P|^2} f \leqslant \alpha f$ holds for some $\alpha > 0$. Assume that there exists $z_0 \in \mathbb{T}$, $z_0 \neq 1$, where $f(z_0) = 0$. Then the assumptions on f imply that $T_{|P|^2} f(z_0) = 0$, which can only be satisfied, due to positivity constraints and assumptions on P, if there exists $z_1 \in \mathbb{T}$ with $z_1^2 = z_0$ and $f(z_1) = 0$. By repeating this argument, we obtain a sequence $\{z_j\}$ of zeros of f, which must form a nontrivial cycle (see Proposition 1). We then show that $P(-z_j) = 0$ follows for all elements of this cycle, which is a contradiction to Cohen's condition. This contradicts to the assumption that f has a zero.

Conversely, let $P(z)$ and $P(-z)$ have a common zero $z_0 \in \mathbb{T}$. It is clear that $z_0 \neq \pm 1$. The function

$$f(z) := (2 - z - 1/z)^k \left(z - z_0^2\right)\left(1/z - z_0^2\right)\left(z - 1/z_0^2\right)\left(1/z - 1/z_0^2\right)$$

is in $P_{N,2k}$ and has double zeros at z_0^2 and its complex conjugate $1/z_0^2$. It is relatively simple to find a constant $\alpha > 0$ such that $T_{|P|^2} f \leqslant \alpha f$. Similarly, the construction of $f \in P_{N,2k}$ with double zeros in a nontrivial cycle can be performed in the case, where Cohen's condition is not satisfied. □

Based on the Perron–Frobenius theory, but with stronger assumptions on P regarding common zeros in $\mathbb{C} \setminus \{0\}$, the following result is shown in [23].

Theorem D. *Let the assumptions of Theorem* 5 *be satisfied. Then the spectral radius of the transfer operator $T_{|P|^2}$ restricted to E_N is* 1, *and $\lambda = 1$ is a simple eigenvalue of $T_{|P|^2}$ with strictly positive eigenfunction $B_\phi \in E_N$. All other eigenvalues of $T_{|P|^2}$ have absolute value less than one.*

We need the following modification to this result which is a direct consequence of the irreducibility of the transfer operator and Theorem D.

Theorem 6. *Let the assumptions in Theorem* 5 *be satisfied. For each $1 \leqslant k \leqslant m$, there exists an eigenfunction f_k of the transfer operator $T_{|P|^2}$ which is strictly positive on $\mathbb{T} \setminus \{1\}$ and has a zero of exact order $2k$ at* 1. *Furthermore, the corresponding eigenvalue is simple, positive and less than one.*

Proof. The existence of an eigenfunction in the interior of the cone $P_{N,2k}$ follows from [24, Theorem 6]. The corresponding eigenvalue is the spectral radius of the restriction of $T_{|P|^2}$ to the subspace $E_{N,2k}$. It is strictly positive, as stated in the same theorem. Theorem 4.3 in [35] assures that the spectral radius is a simple eigenvalue. (We point out even more is true: any other eigenvalue of the same modulus is also simple.) Finally, we infer from Theorem D that the spectral radius must be less than 1, as any eigenfunction of $T_{|P|^2}$ for eigenvalue 1 is non-zero at $z = 1$. □

We are now ready to complete the proof of Theorem 5.

Proof of Theorem 5. By Lemma 5, we can select an eigenfunction $F_m \in E_{N,2m}$ of $T_{|P|^2}$ with associated eigenvalue $0 < \lambda < 1$ which is strictly positive on $\mathbb{T} \setminus \{1\}$ and has a zero of exact order $2m$ at 1. The conditions on P also assure that $B_\phi > 0$ on $\mathbb{T}$. For any $\beta > 1$ we can choose a Laurent polynomial S, by trigonometric approximation, so that

$$\frac{1}{B_\phi + \beta F_m} \leqslant S \leqslant \frac{1}{B_\phi + F_m} \quad \text{on } \mathbb{T}. \tag{5.24}$$

Obviously, S is strictly positive, and the inequalities in (5.24) can be rewritten as

$$\beta F_m \geqslant \frac{1}{S} - B_\phi \geqslant F_m.$$

Since F_m is an element of $E_{N,2m}$, this shows that

$$\frac{1}{S(z)} - B_\phi(z) = \mathcal{O}\big(|z-1|^{2m}\big) \quad \text{near } z = 1.$$

We have thus found a Laurent polynomial S that is strictly positive on $\mathbb{T}$ and satisfies (4.4). Moreover, the monotonicity of the operator $T_{|P|^2}$ and the fact that B_ϕ is an eigenfunction of $T_{|P|^2}$ for the eigenvalue 1 lead to

$$(id - T_{|P|^2})\left(\frac{1}{S}\right) = (id - T_{|P|^2})\left(\frac{1}{S} - B_\phi\right) \geqslant F_m - T_{|P|^2}(\beta F_m) = (1 - \lambda\beta) F_m.$$

The last expression is non-negative for all values of $1 < \beta < 1/\lambda$. Therefore, we obtain

$$\frac{1}{S(z^2)} \geqslant \frac{|P(z)|^2}{S(z)} + \frac{|P(-z)|^2}{S(-z)}, \quad z \in \mathbb{T}.$$

Multiplication by the factor $S(z)S(-z)S(z^2)$ gives (5.19). This completes the proof of Theorem 5. □

Remark 9. In all of the examples in Section 4, straightforward computation of the function S by solving (4.4) with linear algebra methods leads to a matrix $\mathcal{M}_0$ which is positive definite on $\mathbb{T}$. No examination of the spectrum of $T_{|P|^2}$ is needed in these cases. The construction of tight frames with two generators for the cardinal B-splines of order m, $2 \leqslant m \leqslant 4$, relies on this definiteness of $\mathcal{M}_0$. In [14] it is shown that the function S in (4.11) leads to a positive definite matrix $\mathcal{M}_0$ for all $m \geqslant 1$.

Both results in Theorems 4 and 5 can be combined to give the following general result.

Theorem 7. *Let ϕ be a compactly supported refinable function that satisfies* (2.3a)–(2.3c). *If the Laurent polynomials $P(z)$ and $P(-z)$, with P in* (2.2), *have no common roots and P satisfies Cohen's condition, then there exists a tight frame of L^2 with two generators $\psi_1, \psi_2 \in V_1$ that have compact support and m vanishing moments.*

Proof. We summarize the steps of the construction of the tight frame briefly. Theorem 5 gives a VMR Laurent polynomial function S such that $\mathcal{M}$ in (4.2) is positive semidefinite on $\mathbb{T}$. The reduced matrix $\mathcal{M}_0$ in (4.9) and its polyphase decomposition

$$\mathcal{Q}(z^2) := \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} \mathcal{M}_0(z) \begin{bmatrix} 1 & 1/z \\ 1 & -1/z \end{bmatrix}$$

are positive semidefinite as well. The matrix coefficients are Laurent polynomials in z^2, as indicated by the above notation. The matrix Riesz Lemma, namely Theorem 4, provides a factorization

$$\mathcal{Q}\big(z^2\big) = \mathcal{R}\big(1/z^2\big)\mathcal{R}\big(z^2\big).$$

Combination of these steps leads to the two-scale symbols

$$Q_1(z) := \frac{1}{2}\left(\frac{1-z}{2}\right)^m \left[R_{11}(z^2) + zR_{12}(z^2)\right],$$
$$Q_2(z) := \frac{1}{2}\left(\frac{1-z}{2}\right)^m \left[R_{21}(z^2) + zR_{22}(z^2)\right],$$

which define the two generators ψ_1 and ψ_2 of a tight frame with m vanishing moments. □

It should be noted that the result in Theorem 7 does not include assertions about symmetry or inter-orthogonality of ψ_1 and ψ_2. The sibling frames in Theorems 2 and 3 may provide an alternative for situations where any of these properties is required.

6. Sibling and tight frames with one generator

In this section, we consider the particular case of a pair of sibling frames $\{\psi\}$, $\{\tilde{\psi}\}$ with only one generator in V_1. We will show that under certain assumptions on ϕ (stability of integer shifts) sibling frames with one generator can be renormalized to provide tight frames derived from a quadrature mirror filter $\tilde{P}$, see Theorem 8. In particular, if ϕ is a cardinal B-spline of order $m \geqslant 2$, we show that there do not exist compactly supported sibling frames with one generator.

For an arbitrary pair of generators $\{\psi\}$ and $\{\tilde{\psi}\}$, the matrix relation (4.2) becomes

$$\mathcal{M}(z) = \begin{bmatrix} \widetilde{Q}(1/z) \\ \widetilde{Q}(-1/z) \end{bmatrix} \begin{bmatrix} Q(z) & Q(-z) \end{bmatrix}. \tag{6.1}$$

Here, $S(z) = R(z)/T(z)$ is the quotient of two Laurent polynomials with real coefficients, as in Theorem 1, such that $S(1) = 1$.

The rank of $\mathcal{M}$ in (6.1) is at most 1. Therefore, its determinant must vanish identically. This gives

$$\Delta(z^2) = S(z)S(-z) - S(z^2)\left[S(-z)\left|P(z)\right|^2 + S(z)\left|P(-z)\right|^2\right] = 0 \tag{6.2}$$

for all $z \in \mathbb{C} \setminus \{0\}$. This simple observation leads to the following result.

Lemma 6. *Let S be the VMR function of a pair of compactly supported sibling frames $\{\psi\}$, $\{\tilde{\psi}\}$. If S is a Laurent polynomial, then the equations*

$$S(-z)\left|P(z)\right|^2 + S(z)\left|P(-z)\right|^2 = S(-1), \tag{6.3}$$
$$S(z)S(-z) = S(-1)S(z^2), \tag{6.4}$$

are satisfied for all $z \in \mathbb{C} \setminus \{0\}$. In particular, $S(-1) \neq 0$.

Proof. If $S(z)$ is a Laurent polynomial, then $R(z) := z^j S(z)$ is an algebraic polynomial for some $j \in \mathbb{Z}$, and $R(0) \neq 0$. Multiplication of (6.2) by z^{2j} gives an identity where the degrees of the polynomials $z^{2j}S(z)S(-z)$ and $z^{2j}S(z^2)$ agree. Hence, the Laurent polynomial inside the brackets in (6.2) must be constant. For $z = 1$, Eq. (6.2) shows that this constant is $S(-1)$, and we have established (6.3) and (6.4). Since S is non-trivial, $S(-1)$ cannot be zero by virtue of (6.4). □

The following conclusion about the structure of S can be drawn from Lemma 6.

Lemma 7. *Let S be a Laurent polynomial with real coefficients and $S(1) = 1$, such that* (6.3) *and* (6.4) *are satisfied. Then the following statements hold*:

(a) *All roots of S lie on $\mathbb{T}$. The set of all roots is a finite disjoint union of nontrivial cycles. Moreover, all roots in a specific cycle have the same multiplicity.*
(b) *S is real and nonnegative on $\mathbb{T}$; in particular, all roots of S have even multiplicity. Moreover, $S(-1)$ is positive.*

Proof. We use similar arguments as in the proof of Proposition 1. Let S satisfy the assumptions of the lemma. If S is constant in $\mathbb{C}$, we have $S \equiv 1$, and properties (a) and (b) follow immediately.

Let us assume that S is not constant. We have $S(-1) \neq 0$; otherwise S would be the zero constant by (6.4). For the proof of part (a), let $w \in \mathbb{C} \setminus \{0\}$ be a root of S. If we insert $z^2 = w$ into the right-hand side of (6.4), we may conclude that there exists $w_1 \in \mathbb{C}$ such that $w_1^2 = w$ and $S(w_1) = 0$. By repeating this argument we obtain a set of roots

$$F_w := \{w_k\colon\ S(w_k) = 0,\ w_k^{2^k} = w,\ k \geqslant 0\}.$$

This set must be finite and does not contain 1, due to the assumption that S is a Laurent polynomial and $S(1) = 1$. Therefore, there is a non-trivial cycle $\{w_k, w_{k-1}, \ldots, w_{k-m}\} \subset F_w$. This cycle contains w, because $w = w_k^{2^k}$, and therefore the cycle agrees with the set F_w. Hence, we have shown that every root of S is the member of a nontrivial cycle on $\mathbb{T}$. Clearly, there can only be a finite number of such cycles, and distinct cycles must be disjoint. This confirms the first two assertions of part (a).

In order to analyze the multiplicity of all the roots of S in a fixed cycle F, let $w \in F$ be a root with maximal multiplicity among all elements of F. It is a simple fact that $-w$ cannot be an element of any nontrivial cycle on $\mathbb{T}$. Therefore, equation (6.4) implies that w^2 has the same multiplicity as w. This argument can be repeated and assures that all elements of the cycle F have the same multiplicity as w. This completes the proof of part (a).

For part (b) of the lemma, we first show that S is real on $\mathbb{T}$. By assumption, S has real coefficients and all its roots lie on $\mathbb{T}$. Furthermore, as a consequence of part (a), neither 1 nor -1 is a root of S. By Lemma 1, S has a factorization

$$S(z) = s_0 z^\ell S_0(z),$$

where $s_0 \in \mathbb{C}$, ℓ is an integer, and S_0 is a Laurent polynomial with real coefficients which is real on $\mathbb{T}$. Eq. (6.3) can be written as

$$s_0 z^\ell \big((-1)^\ell \big|P(z)\big|^2 S_0(-z) + \big|P(-z)\big|^2 S_0(z)\big) = s_0(-1)^\ell S_0(-1).$$

It follows that z^ℓ must be real for all $z \in \mathbb{T}$, so that $\ell = 0$, and $S(z)/s_0$ is real-valued on $\mathbb{T}$. Finally, our assumption that $S(1) = 1$ implies that S is real-valued on $\mathbb{T}$.

It remains to show that S is nonnegative on $\mathbb{T}$, because the roots of S must then have even multiplicity and $S(-1) > 0$ holds. By continuity of S, it is sufficient to prove that S is strictly positive on the dense set of points

$$z_{j,\ell} := \mathrm{e}^{i2\pi\ell/2^j}, \quad j \geqslant 1,\ 0 \leqslant \ell \leqslant 2^j - 1. \tag{6.5}$$

We first consider the value $S(-1)$. Let w be the zero of S with the smallest positive argument, and $u = \sqrt{w}$ be the element on the smaller circular arc connecting 1 and w. Continuity of the real-valued function S on this arc and $S(1) = 1$ give $S(u) > 0$. Eq. (6.4) requires that

$$S(u)S(-u) = S(w)S(-1) = 0, \quad \text{hence } S(-u) = 0.$$

Inserting this into equation (6.3) gives

$$S(-1) = S(u)\big|P(-u)\big|^2 > 0. \tag{6.6}$$

The strict inequality is justified since $S(-1)$ is nonzero by (6.4).

The positivity of S at all points of the form (6.5) is shown by mathematical induction. We already showed that the assertion is true for $j = 1$; in other words, $S(1) > 0$ and $S(-1) > 0$. Let us assume that all of the values $S(z_{j,\ell})$ are positive, $0 \leqslant \ell < 2^j$. We take any $u := z_{j+1,\ell}$ with $0 \leqslant \ell < 2^j$; this is a complex number on the upper half circle. Eq. (6.4) gives

$$S(u)S(-u) = S(-1)S\big(u^2\big) = S(-1)S(z_{j,\ell}) > 0.$$

The last inequality follows from the induction hypothesis and (6.6). Therefore, $S(u)$ and $S(-u)$ have the same sign. In order to satisfy equation (6.3) both cannot be negative, so that $S(u) > 0$ and $S(-u) > 0$. This proves the positivity of S at all $z_{j+1,\ell}$, and by the induction hypothesis we have positivity at all points (6.5). This completes the proof of the lemma. □

We will next discuss certain consequences of the previous results. Lemma 6 can be used to rewrite the matrix $\mathcal{M}$ in (4.2). First observe that, based on Lemma 6, we have

$$\begin{aligned} S(z) - S(z^2)\left|P(z)\right|^2 &= S(z) - \frac{S(z)S(-z)|P(z)|^2}{S(-1)} \\ &= \frac{1}{S(-1)} S(z)\left(S(-1) - S(-z)\left|P(z)\right|^2\right) \\ &= \frac{1}{S(-1)} S(z)^2\left|P(-z)\right|^2, \end{aligned}$$

and

$$-S(z^2)\overline{P(z)}P(-z) = -\frac{1}{S(-1)} S(z)S(-z)\overline{P(z)}P(-z).$$

This gives

$$\begin{aligned} \mathcal{M}(z) &= \frac{1}{S(-1)} \begin{bmatrix} S(z)^2\left|P(-z)\right|^2 & -S(z)S(-z)\overline{P(z)}P(-z) \\ -S(z)S(-z)P(z)\overline{P(-z)} & S(-z)^2\left|P(z)\right|^2 \end{bmatrix} \\ &= \frac{1}{S(-1)} \begin{bmatrix} z^{-1}S(z)P(-z) \\ -z^{-1}S(-z)P(z) \end{bmatrix} \left[zS(z)\overline{P(-z)} \quad -zS(-z)\overline{P(z)} \right]. \end{aligned}$$

Since S is real and $S(-1) > 0$, we may choose the symbol

$$Q_t(z) = \frac{z}{\sqrt{S(-1)}}\, S(z)P(-1/z) \tag{6.7}$$

in order to obtain a symmetric factorization of $\mathcal{M}$. Therefore we can replace the factorization (6.1) that defines the pair of sibling frames $\{\psi\}$, $\{\tilde{\psi}\}$ with a symmetric factorization that, in turn, defines a tight frame. This is summarized as follows.

Theorem 8. *Let $\{\psi\}$, $\{\tilde{\psi}\}$ be a pair of compactly supported sibling frames associated with a VMR function S. If S is a Laurent polynomial, then the function $\psi_t \in V_1$ with two-scale symbol Q_t in* (6.7) *defines a tight frame of L^2 which is associated with the same VMR function S.*

Remark 10. The result of Theorem 8 can also be expressed in terms of a "renormalization" of the refinable function ϕ. If S is non-negative on $\mathbb{T}$ and satisfies (6.3) and (6.4), Lemma 7 can be used to define the Laurent polynomial U such that $U(z) = U(1/z)$ and $U(z)^2 = S(z)$. (We take half of the multiplicity of all the zeros of S to define the zeros of U.) The new refinable function ϕ_U, defined by

$$\hat{\phi}_U(\xi) := U(z^2)\hat{\phi}(\xi), \quad z = \mathrm{e}^{-i\xi/2},$$

is a finite linear combination of integer shifts of ϕ. Its two-scale symbol takes on the form

$$P_U(z) = \frac{U(z^2)}{U(z)} P(z) = \frac{U(-z)}{\sqrt{S(-1)}} P(z).$$

Eq. (6.3) implies that P_U is a QMF; i.e., we have

$$\left|P_U(z)\right|^2 + \left|P_U(-z)\right|^2 = 1, \quad z \in \mathbb{T}.$$

The tight frame ψ_t in Theorem 8 results from the typical construction based on the QMF; the two-scale symbol of ψ_t relative to the refinable function ϕ_U is $Q_U(z) = zP_U(-1/z)$. In other words, compactly supported sibling frames with one generator and VMR Laurent polynomial S are essentially tight frames defined for a refinable function ϕ_U whose two-scale symbol is a quadrature mirror filter.

Let us end this section by including a discussion of the case where the integer shifts of ϕ form a Riesz basis of V_0. Recall from Proposition 1 that S must be a Laurent polynomial in this case. Therefore, as a consequence of Theorem 8, we have the following.

Corollary 2. *Suppose that the integer shifts of ϕ form a Riesz basis of V_0. Then there exists a pair of compactly supported sibling frames with one generator in V_1 if and only if there exists a compactly supported tight frame with one generator in V_1.*

Hence, it is reasonable to say that compactly supported sibling frames with one generator associated with stable refinable functions are essentially tight frames.

A simple, but important negative conclusion can also be drawn from Theorem 8 as follows.

Theorem 9. *Suppose that the integer shifts of ϕ form a Riesz basis of V_0 and $|P(i)| \neq \sqrt{2}/2$. Then there does not exist a pair of compactly supported sibling frames, and particularly, a tight frame, with one generator in V_1.*

Proof. If there exists a pair of compactly supported sibling frames $\{\psi\}$, $\{\tilde{\psi}\}$ with generators in V_1, then there must be a Laurent polynomial S which satisfies Eqs. (6.3) and (6.4). Note that P and S have real coefficients, and S is real on $\mathbb{T}$. Therefore, $S(i) = S(-i)$ and Eq. (6.4) give

$$S(i)S(-i) = S(i)^2 = S(-1)^2.$$

By Lemma 7, S is non-negative. We thus have $S(i) = S(-i) = S(-1)$. Inserting these values into (6.3) leads to

$$\left|P(i)\right|^2 + \left|P(-i)\right|^2 = 2\left|P(i)\right|^2 = 1.$$

This confirms the result of Theorem 9. □

Note that the value $|P(i)| = \sqrt{2}/2$ is compulsory for every quadrature mirror filter P. On the other hand, there are many examples of stable refinable functions for which $|P(i)|$ does not have this precise value. Any cardinal B-spline N_m of order $m \geqslant 2$, for example, has the property $|P(i)| = 2^{-m/2}$. Therefore, Theorem 9 shows that there do not exist pairs of compactly supported sibling frames and particularly tight frames with one generator which are finite linear combinations of B-splines $N_m(2\cdot - k)$ for $m \geqslant 2$.

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Appendix A

In the following, we give a precise description of the matrix factorization technique used for matrices $\mathcal{M}_0$ in (4.9) whose determinant has low degree.

Proposition 3. *Assume that ϕ is a refinable function with two-scale Laurent polynomial symbol $P(z) = ((1+z)/2)^m P_0(z)$ and $P_0(z) = P_0(-z)$. Let S be a vanishing-moment recovery function that is real-valued on $\mathbb{T}$. Suppose that the matrix $\mathcal{M}_0$ in* (4.9) *is positive* *definite for all $z \in \mathbb{T}$. Let ℓ denote the maximal length of the coefficient sequences of the entries A and B in this matrix, and*

$$\Delta_0(z) = \sum_{k=0}^{r_\Delta} d_k (z + 1/z)^{2k}, \quad r_\Delta \geqslant 0,$$

be its determinant.

(a) *If $r_\Delta = 0$ or 1, then there exists a tight affine frame generated by two compactly supported functions $\{\psi_1, \psi_2\} \in V_1$ that have vanishing moments of order m. The lengths of their symbols Q_i, $i = 1, 2$, is bounded above by $m + \lfloor \ell/2 \rfloor + 2r_\Delta + 1$. If $r_\Delta = 0$, both generators can be chosen to be symmetric (for even m) or antisymmetric (for odd m), provided that ϕ is symmetric.*

(b) *If $r_\Delta = 1$, then there exists a pair of compactly supported sibling frames $\{\psi_1, \psi_2\}$ and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ such that all of the four generators have vanishing moments of order m. The lengths of their symbols Q_i and $\widetilde{Q}_i$, $i = 1, 2$, are bounded above by $m + \lfloor \ell/2 \rfloor + 5$. All generators ψ_i and $\tilde{\psi}_i$, $i = 1, 2$, can be chosen to be symmetric (for even m) or antisymmetric (for odd m), provided that ϕ is symmetric.*

Proof. We first note that the assumptions on the two-scale symbol P imply that the matrix $\mathcal{M}_0$ in (4.9) has the form

$$\mathcal{M}_0(z) = \begin{bmatrix} A(z) & B(z) \\ B(z) & A(-z) \end{bmatrix},$$

where we define $B(z) = -S(z^2) P_0(1/z) P_0(z)$. Moreover, all the entries of this matrix are real-valued on $\mathbb{T}$, by virtue of our assumption on S and the factorization in (4.6). Therefore, A and B can be written as polynomials in $u = (z + z^{-1})/2$ (which is a real variable in $[-1, 1]$), yielding the form

$$A(z) = \sum_{k=0}^{r_A} a_k u^k, \qquad B(z) = \sum_{k=0}^{r_B} b_k u^k, \quad r_A, r_B \in \mathbb{N}. \tag{A.1}$$

Furthermore, all odd coefficients b_{2k+1} are zero due to the assumptions on P_0. This shows that the determinant $\Delta_0(z)$ is a polynomial in u^2.

The polyphase decomposition for the wavelet symbols in (4.10) is achieved by matrix multiplication

$$\begin{aligned}
\widetilde{\mathcal{M}}_0(z) &:= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} \mathcal{M}_0(z) \begin{bmatrix} 1 & 1/z \\ 1 & -1/z \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} A(z) + A(-z) + 2B(z) & z^{-1}\big(A(z) - A(-z)\big) \\ z\big(A(z) - A(-z)\big) & A(z) + A(-z) - 2B(z) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \alpha\big(z^2\big) & \beta\big(z^2\big) \\ z^2 \beta\big(z^2\big) & \gamma\big(z^2\big) \end{bmatrix}.
\end{aligned}$$

The symmetric form (A.1) leads to the representations

$$\alpha\big(z^2\big) = \sum_{k=0}^{r_a} (a_{2k} + b_{2k}) u^{2k} =: a\big(u^2\big),$$

$$\gamma\big(z^2\big) = \sum_{k=0}^{r_c} (a_{2k} - b_{2k}) u^{2k} =: c\big(u^2\big), \quad \text{and}$$

$$\beta\big(z^2\big) = \frac{u}{z} \sum_{k=0}^{r_b} a_{2k+1} u^{2k} =: \frac{u}{z} b\big(u^2\big),$$

where the leading coefficients with index r_a, r_b, and r_c, respectively, are supposed to be non-zero. By using $t = u^2$ we obtain

$$\widetilde{\mathcal{M}}_0(z) = \frac{1}{2} \begin{bmatrix} a(t) & (u/z) b(t) \\ (uz) b(t) & c(t) \end{bmatrix}.$$

Note that $u/z = 1 + 1/z^2$ and $uz = 1 + z^2$ are Laurent polynomials in even powers of z, and one is obtained from the other by substitution of $1/z$ for z. This substitution leaves all the other entries a, b, and c unchanged, so that we have $\widetilde{\mathcal{M}}_0(z) = \widetilde{\mathcal{M}}_0(1/z)^{\mathrm{T}}$.

The determinant of the matrix product is

$$a(t)c(t) - tb^2(t) = \Delta_0\big(z^2\big). \tag{A.2}$$

It is a positive polynomial in the real variable $t \in [0,1]$, by the assumptions of the proposition, and its degree r_Δ is either 0 or 1 depending on cases (a) or (b) in the proposition. We will apply the Euclidean algorithm to reduce the sum of degrees $r_a + r_c$ of the diagonal entries of $\mathcal{M}_0$ to match r_Δ and to make the non-diagonal entries zero. (Note that the positivity of the determinant excludes the possibility of having zero polynomials a or c in the diagonal.) Assume that

$$r_a + r_c > r_\Delta \quad \text{and} \quad r_b \geqslant 0.$$

Then the leading coefficients in the expansion of the determinant must cancel, which gives

$$r_a + r_c = 2r_b + 1.$$

This shows that either $r_a \leqslant r_b < r_c$ or $r_c \leqslant r_b < r_a$ must be satisfied. Let us consider the first case. (The second case can be treated analogously.) There is a polynomial k_1 so that

$$b(t) = k_1(t)a(t) + b_1(t), \qquad r_{b_1} := \deg(b_1) < r_a,$$

where deg:=degree of. Elementary computations lead to

$$\widetilde{\mathcal{M}}_1(z) := \begin{bmatrix} 1 & 0 \\ -(uz)k_1(t) & 1 \end{bmatrix} \widetilde{\mathcal{M}}_0(z) \begin{bmatrix} 1 & -(u/z)k_1(t) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a(t) & (u/z)b_1(t) \\ (uz)b_1(t) & c_1(t) \end{bmatrix},$$

where $c_1 = c - 2k_1tb + k_1^2ta$. Note that the structure of the new matrix $\widetilde{\mathcal{M}}_1$ is the same as before. Moreover, the matrix K_1 on the left of $\widetilde{\mathcal{M}}_0$ and the factor on the right are related by transposition and substitution of $1/z$ for z. In particular, c_1 is a polynomial of $t = u^2$ of degree r_{c_1}, and the determinant has not changed. If $b_1 \equiv 0$, we have reached a situation of a diagonal matrix and proceed to the last step of the construction. If $b_1 \not\equiv 0$, we can show that

$$r_{c_1} < r_a < r_c. \tag{A.3}$$

This means that the sum of the degrees, $r_a + r_c$, was reduced, and an inductive argument will follow. Relation (A.3) is a consequence of the properties

$$r_\Delta = \deg\big(ac_1 - tb_1^2\big) \leqslant 1, \quad 0 \leqslant r_{b_1} < r_a.$$

In the next step, a further reduction is obtained by finding a polynomial $k_2(t)$ so that

$$b_1(t) = k_2(t)c_1(t) + b_2(t), \qquad r_{b_2} := \deg(b_2) < r_{c_1}.$$

In this case, the matrix product

$$\widetilde{\mathcal{M}}_2(z) := \begin{bmatrix} 1 & -(u/z)k_2(t) \\ 0 & 1 \end{bmatrix} \widetilde{\mathcal{M}}_1(z) \begin{bmatrix} 1 & 0 \\ -(uz)k_2(t) & 1 \end{bmatrix} = \begin{bmatrix} a_2(t) & (u/z)b_2(t) \\ (uz)b_2(t) & c_1(t) \end{bmatrix}$$

has the diagonal entry $a_2 = a - 2k_2tb_1 + tk_2^2c_1$ has $\deg(r_{a_2})$, with

$$r_{a_2} < r_{c_1} < r_a.$$

By repeating this procedure finitely many times, we obtain a diagonal matrix

$$\widetilde{\mathcal{M}}_\nu(t) = \begin{bmatrix} a_\nu(t) & 0 \\ 0 & c_\nu(t) \end{bmatrix}. \tag{A.4}$$

The matrices in the Euclidean algorithm which appear on the left of $\widetilde{\mathcal{M}}_0$ and $\widetilde{\mathcal{M}}_1$, etc., constitute a matrix

$$R(z^2) = K_\nu(t)\cdots K_1(t) = \begin{bmatrix} R_{11}(t) & (u/z)R_{12}(t) \\ (uz)R_{21}(t) & R_{22}(t) \end{bmatrix}.$$

This is clearly a matrix with determinant 1 whose entries R_{ik} are Laurent polynomials of z^2. All matrices with this particular structure define a ring, and the factors K_j that constitute R are invertible elements of this ring. Therefore, R is invertible and

$$R^{-1}(z^2) = \begin{bmatrix} R_{22}(t) & -(u/z)R_{12}(t) \\ -(uz)R_{21}(t) & R_{11}(t) \end{bmatrix}.$$

For degree considerations, we define

$$\lambda := \max\{\deg(R_{11}), \deg(R_{12}), \deg(R_{21}), \deg(R_{22})\},$$

where the polynomials are considered in the variable $t = u^2 = (z+1/z)^2$. It can be shown that $\lambda \leqslant 1/2 \max\{r_a, r_c\}$ where these two numbers denote the degree of the diagonal entries of $\widetilde{\mathcal{M}}_0$. Hence,

$$4\lambda \leqslant 2\max\{r_a, r_c\} \leqslant \lfloor \ell/2 \rfloor,$$

with ℓ as in the proposition.

Let us now consider the final decomposition step. The determinant of the diagonal matrix $\widetilde{\mathcal{M}}_\nu$ is Δ_0, and the matrix is positive definite for all $t \in [0,1]$. If Δ_0 is constant, then both diagonal elements are positive constants. The trivial factorization

$$\widetilde{\mathcal{M}}_\nu(t) = \begin{bmatrix} \sqrt{a_\nu} & 0 \\ 0 & \sqrt{c_\nu} \end{bmatrix}^2$$

is used to find a symmetric factorization

$$\widetilde{\mathcal{M}}_0(z) = R^{-1}(z^2) \begin{bmatrix} \sqrt{a_\nu} & 0 \\ 0 & \sqrt{c_\nu} \end{bmatrix}^2 R^{-1}(1/z^2)^{\mathrm{T}}. \tag{A.5}$$

The generators for the tight frame can thus be defined through their two-scale symbols

$$Q_1(z) = \sqrt{a_\nu}\big((1-z)/2\big)^m \big[R_{22}(t) - (z+1/z)R_{21}(t)\big],$$
$$Q_2(z) = \sqrt{c_\nu}\, z\big((1-z)/2\big)^m \big[R_{11}(t) - (z+1/z)R_{12}(t)\big],$$

with $t = (z+1/z)^2$. These symbols are even or odd depending on the parity of m. Hence, the symmetry or antisymmetry of the functions ψ_1 and ψ_2 is assured, provided that ϕ is symmetric. The length of the coefficient sequences for the symbols is bounded by $m + 4\lambda + 1$. This number is bounded above by $m + \lfloor \ell/2 \rfloor + 1$, as claimed in the proposition.

If $r_\Delta = 1$, then one diagonal entry of $\widetilde{\mathcal{M}}_\nu$ is constant and the other is a linear polynomial in t. Let us assume that $a_\nu(t)$ has degree 1. By the Riesz Lemma, we can find a factorization

$$a_\nu(t) = c_0 + c_1(z^2 + 1/z^2) = (\gamma_0 + \gamma_1 z^2)(\gamma_0 + \gamma_1/z^2).$$

Using the same matrix $R(z^2)$ as above, this gives rise to the definition of tight frames with the two-scale symbols

$$Q_1(z) = (\gamma_0 + \gamma_1 z^2)\big((1-z)/2\big)^m \big[R_{22}(t) - (z+1/z)R_{21}(t)\big],$$
$$Q_2(z) = \sqrt{c_\nu}\, z\big((1-z)/2\big)^m \big[R_{11}(t) - (z+1/z)R_{12}(t)\big].$$

While the function ψ_2 has the same symmetry properties as outlined before, only special circumstances (such as even multiplicity of the roots of Δ_0) would make the symbol Q_1 symmetric or antisymmetric. The length of the coefficient sequences is seen to be bounded by $m + [\ell/2] + 1$ for Q_2 and $m + [\ell/2] + 3$ for Q_1. This completes the proof of part (a) of the proposition.

In order to achieve symmetry for all generators of the frame, the non-symmetric factorization

$$\widetilde{\mathcal{M}}_\nu(z) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{c_\nu} \end{bmatrix} \begin{bmatrix} a_\nu(t) & 0 \\ 0 & \sqrt{c_\nu} \end{bmatrix}$$

can be used instead. This leads to the definition of sibling frames $\{\psi_1, \psi_2\}$, $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ where $\psi_2 = \tilde{\psi}_2$ is defined with a two-scale symbol Q_2 as above. The functions ψ_1 and $\tilde{\psi}_1$, however, are determined by their two-scale symbols

$$Q_1(z) = \bigl((1-z)/2\bigr)^m \bigl[R_{11}(t) + (z + 1/z) R_{21}(t)\bigr],$$
$$\widetilde{Q}_1(z) = a_\nu(t) Q_1(z).$$

From this definition, it is clear that the two-scale symbols of ψ_1 and $\tilde{\psi}_1$ differ only by a factor that is a Laurent polynomial in z^2. Symmetry of the Laurent polynomials Q_1 and $\widetilde{Q}_1$ is seen exactly as in the previous cases. The lengths of the coefficient sequences are $m + [\ell/2] + 1$ for Q_1, Q_2, and $\widetilde{Q}_2$, and $m + [\ell/2] + 5$ for $\widetilde{Q}_1$. This concludes the proof of the second part of the proposition. $\square$

This type of factorization is used in the examples of Section 4.1.

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