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LETTER TO THE EDITOR

Compactly Supported Tight Frames Associated with Refinable Functions[1]

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Abstract—It is well known that in applied and computational mathematics, cardinal B-splines play an important role in geometric modeling (in computer-aided geometric design), statistical data representation (or modeling), solution of differential equations (in numerical analysis), and so forth. More recently, in the development of wavelet analysis, cardinal B-splines also serve as a canonical example of scaling functions that generate multiresolution analyses of $L^2(-\infty, \infty)$. However, although cardinal B-splines have compact support, their corresponding orthonormal wavelets (of Battle and Lemarie) have infinite duration. To preserve such properties as self-duality while requiring compact support, the notion of tight frames is probably the only replacement of that of orthonormal wavelets. In this paper, we study compactly supported tight frames $\Psi = \{\psi^1, \ldots, \psi^N\}$ for $L^2(-\infty, \infty)$ that correspond to some refinable functions with compact support, give a precise existence criterion of Ψ in terms of an inequality condition on the Laurent polynomial symbols of the refinable functions, show that this condition is not always satisfied (implying the nonexistence of tight frames via the matrix extension approach), and give a constructive proof that when Ψ does exist, two functions with compact support are sufficient to constitute Ψ, while three guarantee symmetry/anti-symmetry, when the given refinable function is symmetric.

1. INTRODUCTION AND RESULTS

This paper is concerned with the study of compactly supported tight frames as a replacement of compactly supported orthonormal (o.n.) wavelets when the system $\{\phi(\cdot - k): k \in \mathbf{Z}\}$ generated by the corresponding compactly supported scaling function ϕ is not orthogonal and, more generally, when ϕ is simply a refinable function (meaning that $\{\phi(\cdot - k): k \in \mathbf{Z}\}$ may not be stable). For simplicity, we only consider the basic univariate $L^2 := L^2(-\infty, \infty)$ setting, with inner product and norm denoted by $\langle\, , \rangle$,

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and $\| \ \|$, respectively. This study is motivated by the recent work of Ron and Shen [12] and the elegant characterizations of o.n. wavelets and those that correspond to some multiresolution analysis (MRA) scaling functions presented in the monograph [9] by Hernández and Weiss.

1.1. The Notion of Minimum-energy Tight Frames

A function $\psi \in L^2$ with $\|\psi\| = 1$ is called an o.n. wavelet if the family

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \qquad j, k \in \mathbf{Z}, \tag{1.1}$$

generated by ψ, constitutes an o.n. basis of L^2. It is well known that o.n. wavelets $\psi \in L^2$ are completely characterized by the set of conditions

$$\begin{cases} \|\psi\| = 1, & \\ \sum_{j\in\mathbf{Z}} |\hat{\psi}(2^j\omega)|^2 = 1, & \text{a.e.;} \\ \sum_{j=0}^{\infty} \hat{\psi}(2^j\omega)\overline{\hat{\psi}(2^j(\omega + 2k\pi))} = 0, & \text{a.e., } k \in 2\mathbf{Z}+1, \end{cases} \tag{1.2}$$

in terms of their Fourier transforms (see [9, Theorem 1.1, p. 332]). It is also well known that the characterization of o.n. wavelets in (1.2) does not necessarily imply the existence of an associated scaling function that generates an MRA of L^2.

When an o.n. wavelet $\psi \in L^2$ is associated with some MRA, it is called an MRA wavelet in [9]. Again this subfamily of o.n. wavelets can be completely characterized in terms of their Fourier transforms. For instance, in [9, Theorem 3.2, p. 355], it is proved that $\psi \in L^2$, with $\|\psi\| = 1$, is an MRA wavelet, if and only if it is an o.n. wavelet and satisfies the condition

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbf{Z}} |\hat{\psi}(2^j(\omega + 2k\pi))|^2 = 1, \qquad \text{a.e.} \tag{1.3}$$

Although this characterization is most elegant, it does not reveal the explicit relationship between ψ and the scaling function ϕ that generates the MRA, and furthermore, orthogonal decomposition does not immediately follow from (1.3). To motivate our generalization of the notion of MRA wavelets to that of MRA tight frames, we consider the following two definitions of MRA wavelets that are equivalent under certain mild conditions on the scaling function. The first definition addresses the MRA relationship more explicitly, while the second one is more useful in the discussion of orthogonal wavelet decomposition.

DEFINITION 1. An o.n. wavelet $\psi \in L^2$ is called an MRA wavelet associated with a scaling function $\phi \in L^2$ that generates an MRA $\{V_j\}$, if $\psi \in V_1$.

Here, the standard notation of MRA $\{V_j\}$ is used; namely,

$$V_j := \text{clos}_{L^2}\langle \phi_{j,k} : k \in \mathbf{Z}\rangle, \qquad j \in \mathbf{Z}, \tag{1.4}$$

where the double-index notation in (1.1) is also used for ϕ.

DEFINITION 1′. Let $\phi \in L^2$ be an o.n. scaling function that generates an MRA $\{V_j\}$. Then a function $\psi \in V_1$, with $\|\psi\| = 1$, is called an MRA wavelet associated with ϕ, if

$$\sum_{k\in\mathbf{Z}} |\langle f, \phi_{1,k}\rangle|^2 = \sum_{k\in\mathbf{Z}} |\langle f, \phi_{0,k}\rangle|^2 + \sum_{k\in\mathbf{Z}} |\langle f, \psi_{0,k}\rangle|^2, \qquad \text{all } f \in L^2. \tag{1.5}$$

Here, a scaling function ϕ is said to be o.n. if the family of its integer translates $\{\phi(\cdot - k) : k \in \mathbf{Z}\}$ is an o.n. system. We first remark that (1.5) is equivalent to the formulation

$$\sum_{k\in\mathbf{Z}} \langle f, \phi_{1,k}\rangle \phi_{1,k} = \sum_{k\in\mathbf{Z}} \langle f, \phi_{0,k}\rangle \phi_{0,k} + \sum_{k\in\mathbf{Z}} \langle f, \psi_{0,k}\rangle \psi_{0,k}, \tag{1.6}$$

of orthogonal wavelet decomposition of $V_1 = V_0 \oplus W_0$, where

$$W_j := V_{j+1} \ominus V_j, \qquad j \in \mathbf{Z}. \tag{1.7}$$

Also, by replacing the indices 1 and 0 by $j + 1$ and j, respectively, in (1.5) and telescoping over all $j \in \mathbf{Z}$, we have the Parseval identity:

$$\sum_{j,k\in\mathbf{Z}} |\langle f, \psi_{j,k}\rangle|^2 = \|f\|^2, \qquad \text{all } f \in L^2 \tag{1.8}$$

(see [6, pp. 141–143] and observe that $|\hat{\phi}(0)| = 1$ by [9, Theorem l.7, p. 46]). Hence, it follows that Definition 1 and Definition 1′ are equivalent, provided that ϕ is an o.n. scaling function.

The reason for introducing Definition 1′ is to motivate the following notion of minimum-energy frames. First recall that a family $\Psi = \{\psi^1, \dots, \psi^N\} \subset L^2$ is called a tight frame of L^2 if it satisfies

$$\sum_{i=1}^{N} \sum_{j,k\in\mathbf{Z}} |\langle f, \psi^i_{j,k}\rangle|^2 = \|f\|^2, \qquad \text{all } f \in L^2. \tag{1.9}$$

Here, for convenience, we have normalized ψ^i by the same constant so that the frame bound in (1.9) is equal to 1. The generalization of the notion of o.n. wavelets from (1.8) to that of tight frames in (1.9) is obvious. The important differentiation is that for o.n. wavelets, ψ in (1.8) must have L^2-norm equal to 1. To address the relation of a tight frame associated with some refinable function ϕ which generates the nested subspaces $\{V_j\}_{j=-\infty}^{\infty}$ defined in (1.4) and which approximates L^2, namely

$$0 \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \to L^2, \tag{1.10}$$

in the sense that

$$\operatorname{clos}_{L^2} \bigcup_{j\in\mathbf{Z}} V_j = L^2, \tag{1.11}$$

we generalize the above two (equivalent) definitions as follows. Here, we emphasize that $\{\phi(\cdot - k) : k \in \mathbf{Z}\}$ is not necessarily a Riesz basis of V_0.

As a generalization of Definition 1 to tight frames, we consider the following.

DEFINITION 2. A finite family $\Psi = \{\psi^1, \dots, \psi^N\} \subset L^2$ that satisfies (1.9) is called an MRA tight (wavelet-) frame, with frame bound equal to 1, associated with a refinable function ϕ that generates the nested subspaces $\{V_j\}$ of L^2 in the sense of (1.10), if $\Psi \subset V_1$.

As a generalization of Definition 1′, we introduce the following notion of *minimum-energy* (*wavelet*)-*frames* associated with some refinable functions.

DEFINITION 3. Let $\phi \in L^2$, with $\hat{\phi} \in L^\infty$, $\hat{\phi}$ continuous at 0, and $\hat{\phi}(0) = 1$, be a refinable function that generates the nested subspaces $\{V_j\}$ in the sense of (1.10). Then a finite family of functions $\Psi := \{\psi^1, \ldots, \psi^N\} \subset V_1$ is called a minimum-energy (wavelet-) frame associated with ϕ, if

$$\sum_{k\in\mathbf{Z}} |\langle f, \phi_{1,k}\rangle|^2 = \sum_{k\in\mathbf{Z}} |\langle f, \phi_{0,k}\rangle|^2 + \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}} |\langle f, \psi^i_{0,k}\rangle|^2, \qquad \text{all } f \in L^2. \tag{1.12}$$

Remark 1. By telescoping as in (1.5) and (1.8), it follows that a minimum-energy frame according to Definition 3 satisfies (1.9) (see [6, pp. 141–143], using the assumption $\hat{\phi} \in L^\infty$, $\hat{\phi}$ continuous at 0, and $\hat{\phi}(0) = 1$); and hence, a minimum-energy frame Ψ is necessarily a tight frame for L^2, with frame bound equal to 1.

Remark 2. In contrast to the equivalence of Definitions 1 and 1′, for o.n. ϕ, the notion of minimum-energy frames associated with a refinable ϕ is more restrictive than that of MRA tight frames, as can be seen from an example in Ron and Shen [12, Sect. 6].

Again, it is clear that (1.12) is equivalent to the formulation

$$\sum_{k\in\mathbf{Z}} \langle f, \phi_{i,k}\rangle \phi_{1,k} = \sum_{k\in\mathbf{Z}} \langle f, \phi_{0,k}\rangle \phi_{0,k} + \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}} \langle f, \psi^i_{0,k}\rangle \psi^i_{0,k}, \qquad \text{all } f \in L^2. \tag{1.13}$$

The interpretation of minimum energy will be clarified in Section 4.

1.2. Why Minimum-energy Frames?

Let $\phi \in L^2$ be an o.n. compactly supported scaling function governed by a two-scale relation

$$\phi(x) = \sum_k p_k \phi(2x - k) \tag{1.14}$$

for some finite (two-scale) sequence $\{p_k\}$. Then the function

$$\psi(x) := \sum_k (-1)^k \bar{p}_{1-k} \phi(2x - k) \tag{1.15}$$

is a compactly supported o.n. MRA wavelet. Such functions $\phi(x)$ and $\psi(x)$, constructed by Daubechies in [5], are also called Daubechies scaling functions and wavelets, respectively. It was also shown in [5], however, that with the exception of the first order cardinal B-spline and its corresponding Haar function, any compactly supported o.n. scaling function and its corresponding MRA wavelet do not have the symmetry or anti-symmetry property. For this and other reasons, biorthogonal scaling functions and wavelets with compact support were introduced by Cohen *et al.* in [4] by using two different MRAs. One of the disadvantages of this biorthogonal approach is that since two different MRAs are used, the analysis and synthesis operations of the biorthogonal wavelet pair $(\psi, \tilde{\psi})$ cannot be interchanged at any particular scale 2^{j_0}, say. In other words, "change-of-bases" between $\{\psi_{j_0,k} : k \in \mathbf{Z}\}$ and $\{\tilde{\psi}_{j_0,k} : k \in \mathbf{Z}\}$ is not possible.

To demonstrate the importance of the feature of change-of-bases at any scale, let us consider the mth order cardinal B-spline $N_m(x)$, $m \geq 2$, defined inductively by

$$N_m(x) := \int_0^1 N_{m-1}(t-x)\,dt, \tag{1.16}$$

with $N_1(x)$ denoting the characteristic function of the unit interval $[0,1]$, along with its corresponding B-wavelet

$$\psi_m(x) = \sum_{k=0}^{3m-2} (-1)^k \left[\frac{1}{2^{m-1}} \sum_{l=0}^{m} \binom{m}{l} N_{2m}(k-l+1) \right] N_m(2x-k) \tag{1.17}$$

(see [1, p. 188]). Since $N_m(x)$ and $\psi_m(x)$ are very good approximations both in the time and in the frequency domains of certain Gaussian and (cosine or sine) modulated Gaussian (depending on even or odd m), respectively (see comparisons in [1, pp. 186–187], graphs of $\psi_m(x)$ and $\hat{\psi}_m(\omega)$ in [2, pp. 103–104], and asymptotic results in [2, pp. 114–117] and [3]), the B-wavelets $\psi_m(x)$ are very desirable for both analysis and synthesis. Hence, change of bases between $\{\psi_m(x-k)\}$ and its dual $\{\tilde{\psi}_m(x-k)\}$, so as to use $\psi_m(x)$ both for analysis and synthesis, is needed. (See [2, pp. 129–131] for a discussion of change of bases.)

The challenge is to avoid the complication of change of bases but still to use the same wavelets, both for analysis and for synthesis. Besides o.n. wavelets, minimum-energy frames can serve this purpose well.

1.3. *Characterization of Minimum-energy Frames*

In this section, we give a complete characterization of minimum-energy frames associated with some given refinable functions in terms of their two-scale symbols. For convenience, we only consider symbols in the Wiener class $\mathcal{W}$, meaning that the coefficient sequences of the symbols are in ℓ^1. Let $\phi \in L^2$, with $\hat{\phi} \in L^\infty$, $\hat{\phi}$ continuous at 0, and $\hat{\phi}(0) = 1$, be a refinable function with refinement equation

$$\phi(x) = \sum_{k \in \mathbf{Z}} p_k \phi(2x-k) \tag{1.18}$$

such that its refinement (or two-scale) symbol

$$P(z) := \frac{1}{2} \sum_{k \in \mathbf{Z}} p_k z^k \tag{1.19}$$

is in $\mathcal{W}$. Let $\{V_j\}$ be the nested subspaces generated by ϕ which approximate L^2 in the sense of (1.10), and consider $\Psi = \{\psi^1, \ldots, \psi^N\} \subset V_1$, with

$$\psi^\ell(x) = \sum_{k \in \mathbf{Z}} q_k^\ell \phi(2x-k) \tag{1.20}$$

and two-scale symbols

$$Q_\ell(z) := \frac{1}{2} \sum_{k \in \mathbf{Z}} q_k^\ell x^k \in \mathcal{W}, \qquad \ell = 1, \ldots, N. \tag{1.21}$$

With $P(z)$ and $Q_\ell(z)$, we formulate the $(N+1)\times 2$ matrix

$$\mathcal{R}(z) := \begin{bmatrix} P(z) & P(-z) \\ Q_1(z) & Q_1(-z) \\ \vdots & \vdots \\ Q_N(z) & Q_N(-z) \end{bmatrix}, \tag{1.22}$$

and use the standard notation $\mathcal{R}^*(z)$ to represent the complex conjugate of the transpose of $\mathcal{R}(z)$. The following characterization will be used in this paper to study the existence of minimum-energy frames associated with ϕ and to develop an algorithm to construct these frames when they exist.

LEMMA 1. *Let $P(z)$ and $Q_\ell(z)$, $\ell = 1, \ldots, N$, in* (1.19) *and* (1.21) *be Laurent polynomials that govern the compactly supported refinable function $\phi \in L^2$ and the family $\Psi = \{\psi^1, \ldots, \psi^N\} \subset V_1$. Suppose that $\hat{\phi}(0) = 1$ and that $\{V_j\}$ generated by ϕ satisfies* (1.10). *Then the following statements are equivalent.*

(i) *Ψ is a minimum-energy frame associated with ϕ.*

(ii)

$$\mathcal{R}^*(z)\mathcal{R}(z) = I_2, \qquad \textit{for } |z| = 1. \tag{1.23}$$

(iii)

$$\alpha_{m,\ell} := \sum_{k\in\mathbf{Z}} \left(p_{m-2k}\, p_{\ell-2k} + \sum_{i=1}^{N} q^i_{m-2k} q^i_{\ell-2k} \right) - 2\delta_{m,\ell} \tag{1.24}$$

satisfies

$$\alpha_{m,\ell} = 0 \qquad \textit{all } m, \ell \in \mathbf{Z}, \tag{1.25}$$

where $\delta_{m,\ell}$ is the Kronecker delta symbol.

Our consideration of (1.23) is motivated by a result in Ron and Shen [12] which says that (1.23) is a sufficient condition for the family $\{\psi^i_{j,k} : i = 1, \ldots, N;\ \ j, k \in \mathbf{Z}\}$ to be a tight frame of L^2, with frame bound equal to 1, as in (1.9).

In Lemma 1, that (ii) implies (i) for the case $N = 1$ was first proved in Lawton [10]. Lawton's result was then generalized to the multivariate setting with dilation matrices (cf. [8, 12]).

1.4. What Refinable Functions Generate Minimum-energy Frames?

Since one of the main reasons for studying MRA tight frames is to achieve compact support (for both analyzing and synthesizing wavelets), we consider, in the remaining writing of this paper, as in the statement of Lemma 1, only compactly supported refinable functions ϕ and $\Psi = \{\psi^1, \ldots, \psi^N\} \subset V_1$, so that the symbols $P(z)$ and $Q_1(z), \ldots, Q_N(z)$ are Laurent polynomials. The first main result of this paper is the following.

THEOREM 1. *A compactly supported refinable function $\phi \in L^2$, with $\hat{\phi}(0) = 1$ and two-scale Laurent polynomial symbol $P(z)$, has an associated minimum-energy frame Ψ*

with compact support, if and only if $P(z)$ *satisfies*

$$|P(z)|^2 + |P(-z)|^2 \leq 1, \qquad \textit{all } |z| = 1. \tag{1.26}$$

As an example, let us consider the mth order cardinal B-splines N_m defined in (1.16). It is well known that the two-scale symbol of N_m is

$$P_m(z) = \left(\frac{1+z}{2}\right)^m, \tag{1.27}$$

which clearly satisfies (1.26). Hence, associated with each N_m, we have a minimum-energy frame. We will return to elaborate on this important example in Sections 1.5 and 3.1.

Remark 3. The restriction (1.26) on the two-scale symbol $P(z)$ of a refinable function ϕ is a necessary condition for the existence of an MRA tight frame associated with ϕ via the rectangular unitary matrix extension approach (1.23), even if ϕ is not compactly supported (see the proof of Theorem 1 in Section 2). The reason is that minimum-energy frames are those MRA tight frames constructed via this matrix extension approach (see Lemma 1). This points out an incorrect statement in Ron and Shen [12, Sect. 6], where the authors believe that for $N \geq 2$ in (1.22), there does not seem to be any a priori restriction on $P(z)$ (other than the most basic conditions, such as $P(1) = 1$) for ϕ to have an associated MRA tight frame by the unitary extension principle.

To demonstrate the reality of nonexistence of minimum-energy frames for certain compactly supported refinable functions, let us consider the biorthogonal wavelets of Cohen *et al.* [4], where we use N_m, $m \geq 2$, to generate an MRA $\{V_j\}$, and another compactly supported scaling function $\tilde{\phi}_m \in L^2$, dual to N_m, to generate the dual MRA $\{\tilde{V}_j\}$. By the duality between N_m and $\tilde{\phi}_m$, we have

$$1 = \sum_{k \in \mathbf{Z}} \overline{\hat{N}_m(2\pi k)} \hat{\tilde{\phi}}_m(2\pi k) = \overline{\hat{N}_m(0)} \hat{\tilde{\phi}}_m(0) = \hat{\tilde{\phi}}_m(0). \tag{1.28}$$

On the other hand, the two-scale symbol $\tilde{P}_m(z)$ of $\tilde{\phi}_m$ is related to the two-scale symbol $P_m(z)$ in (1.27) of N_m by

$$\overline{P_m(z)}\tilde{P}_m(z) + \overline{P_m(-z)}\tilde{P}_m(-z) = 1, \qquad |z| = 1. \tag{1.29}$$

Hence, by the Cauchy–Schwartz inequality, we have

$$\begin{aligned} 1 &\leq \left(|P_m(z)\tilde{P}_m(z)| + |P_m(-z)\tilde{P}_m(-z)|\right)^2 \\ &\leq \left(|P_m(z)|^2 + |P_m(-z)|^2\right)\left(|\tilde{P}_m(z)|^2 + |\tilde{P}_m(-z)|^2\right), \end{aligned}$$

and in view of (1.26) for $P_m(z)$, we see that

$$|\tilde{P}_m(z)|^2 + |\tilde{P}_m(-z)|^2 \geq 1, \qquad |z| = 1. \tag{1.30}$$

That $\{\tilde{\phi}_m(\cdot - k)\}$ is not an o.n. system for $m \geq 2$ implies that strict inequality in (1.30) must hold on some subset of $|z| = 1$ with positive measure. Hence, by Theorem 1, there does not exist a minimum-energy frame associated with the scaling function $\tilde{\phi}_m$, for any $m \geq 2$.

1.5. Compactly Supported Minimum-energy Frames with Two Generators

The second main result of this paper is the following.

THEOREM 2. *Let $\phi \in L^2$, with $\hat{\phi}(0) = 1$, be a compactly supported refinable function with two-scale Laurent polynomial symbol $P(z)$ that satisfies*

$$|P(z)|^2 + |P(-z)|^2 \leq 1, \qquad |z| = 1. \tag{1.31}$$

Then there exists a minimum-energy frame $\Psi = \{\psi^1, \psi^2\}$ associated with ϕ, where both ψ^1 and ψ^2 have compact support.

For a cardinal B-spline N_m of arbitrary order $m \geq 2$, there exist two functions

$$\psi_m^\ell(x) = \sum_{k=0}^{n_\ell} q_k^\ell N_m(2x - k), \qquad \ell = 1, 2, \tag{1.32}$$

where n_1 and n_2 are nonnegative integers, such that $\Psi_m = \{\psi_m^1, \psi_m^2\}$ is a minimum-energy (and hence, tight) frame associated with the cardinal B-spline N_m in the sense of (1.12). In Ron and Shen [12], it was shown that, associated with N_m, there is a compactly supported tight frame with m functions. In this regard, it is also stated in Ron and Shen [14, Sect. 2] by an observation of a B-spline bi-frame example that it is possible to derive from N_m a tight compactly supported spline frame with two generators for which one is shifted along integer translations, while the other is shifted along the half-integer translations. This approach, which originated in the construction of Strömberg spline wavelets (see [2, pp. 75–77]), differs from the integer-translate consideration in this paper.

1.6. Compactly Supported and Symmetric Minimum-energy Frames with Three Generators

When the given compactly supported refinable function is symmetric and satisfies (1.26), we show that three generating functions are sufficient to constitute a minimum-energy frame with symmetry/anti-symmetry, as follows.

THEOREM 3. *For any compactly supported symmetric scaling function $\phi \in L^2$ with $\hat{\phi}(0) = 1$ and two-scale Laurent polynomial symbol $P(z)$ satisfying* (1.26), *there exists a compactly supported minimum-energy frame $\Psi = \{\psi^1, \psi^2, \psi^3\}$ associated with ϕ, with symmetric or anti-symmetric ψ^1, ψ^2, and ψ^3.*

1.7. Organization of the Paper

The results stated in this section will be proved in the next section. Examples are given in Section 3, where both cardinal B-splines and interpolating scaling functions will be considered. It will be shown that when the interpolating scaling functions, with two-scale symbols $P_m^I(z)$, are autocorrelations of the mth order Daubechies o.n. scaling functions with two-scale symbols $P_m^D(z)$, then the two-scale symbols of the corresponding tight-frame generators have explicit formulations:

$$Q_m^1(z) = 1 - P_m^I(z) \qquad \text{and} \qquad Q_m^2(z) = \sqrt{2} z P_m^D\left(\frac{1}{z}\right) P_m^D\left(-\frac{1}{z}\right).$$

In Section 4, we will discuss the notion of minimum-energy and, for completeness, write down the frame decomposition and reconstruction algorithms.

2. PROOF OF RESULTS

In this section, we give the proofs of Lemma 1 and Theorems 1–3.

Proof of Lemma 1. First, we observe, by using the two-scale relations (1.18) and (1.20) and the notation in (1.24), that (1.13) can be written as

$$\sum_{\ell\in\mathbf{Z}}\sum_{m\in\mathbf{Z}}\alpha_{m,\ell}\langle f,\phi(2\cdot-m)\rangle\phi(2x-\ell)=0, \qquad \text{all } f\in L^2, \tag{2.1}$$

where $\{\alpha_{m,\ell}\}$ is defined in (1.24). On the other hand, (1.23) can be reformulated as

$$\begin{cases} |P(z)|^2+\sum_{i=1}^N|Q_i(z)|^2=1; \\ P(z)\overline{P(-z)}+\sum_{i=1}^N Q_i(z)\overline{Q_i(-z)}=0, \qquad |z|=1, \end{cases} \tag{2.2}$$

which is equivalent to

$$\begin{cases} P(z)(\overline{P(z)}+\overline{P(-z)})+\sum_{i=1}^N Q_i(z)(\overline{Q_i(z)}+\overline{Q_i(-z)})=1; \\ P(z)(\overline{P(z)}-\overline{P(-z)})+\sum_{i=1}^N Q_i(z)(\overline{Q_i(z)}-\overline{Q_i(-z)})=1, \qquad |z|=1, \end{cases} \tag{2.3}$$

or

$$\begin{cases} P(z)\sum_k p_{-2k}z^{2k}+\sum_{i=1}^N Q_i(z)\sum_k q^i_{-2k}z^{2k}=1; \\ P(z)\sum_k p_{1-2k}z^{2k-1}+\sum_{i=1}^N Q_i(z)\sum_k q^i_{1-2k}z^{2k-1}=1, \qquad |z|=1. \end{cases} \tag{2.4}$$

Following [1, pp. 142–143], we multiply the two identities in (2.4) by $\hat{\phi}(\omega/2)$ and $z\hat{\phi}(\omega/2)$, respectively, where $z=e^{-i\omega/2}$, to get

$$\begin{cases} \hat{\phi}(\omega/2)=\sum_k\left(p_{-2k}z^{2k}P(z)\hat{\phi}(\omega/2)+\sum_{i=1}^N q^i_{-2k}z^{2k}Q^i(z)\hat{\phi}(\omega/2)\right); \\ \hat{\phi}(\omega/2)e^{-i\omega/2}=\sum_k\left(p_{1-2k}z^{2k}P(z)\hat{\phi}(\omega/2)+\sum_{i=1}^N q^i_{1-2k}z^{2k}Q^i(z)\hat{\phi}(\omega/2)\right). \end{cases}$$

Hence, (2.4) is equivalent to

$$\begin{cases} \hat{\phi}(\omega/2)=\sum_k\left(p_{-2k}z^{2k}\hat{\phi}(\omega)+\sum_{i=1}^N q^i_{-2k}z^{2k}\hat{\psi}(\omega)\right); \\ \hat{\phi}(\omega/2)e^{-i\omega/2}=\sum_k\left(p_{1-2k}z^{2k}\hat{\phi}(\omega)+\sum_{i=1}^N q^i_{1-2k}z^{2k}\hat{\psi}(\omega)\right), \end{cases} \tag{2.5}$$

or equivalently,

$$\begin{cases} 2\phi(2x)=\sum_k\left(p_{-2k}\phi(x-k)+\sum_{i=1}^N q^i_{-2k}\psi^i(x-k)\right); \\ 2\phi(2x-1)=\sum_k\left(p_{1-2k}\phi(x-k)+\sum_{i=1}^N q^i_{1-2k}\psi^i(x-k)\right), \end{cases} \tag{2.6}$$

which can be reformulated as

$$\phi(2x-\ell)=\frac{1}{2}\sum_k\left\{p_{\ell-2k}\phi(x-k)+\sum_{i=1}^N q^i_{\ell-2k}\psi^i(x-k)\right\}, \qquad \ell\in\mathbf{Z}. \tag{2.7}$$

By using the two-scale relations (1.18) and (1.20), we can rewrite (2.7) as

$$\sum_{m\in\mathbf{Z}} \alpha_{m,\ell}\phi(2x-m)=0, \qquad \text{all } \ell\in\mathbf{Z}. \tag{2.8}$$

In other words, (1.23) is equivalent to (2.8). Hence, the proof of Lemma 1 reduces to the proof of the equivalence of (2.1), (2.8), and (1.25).

It is obvious that (1.25) $\Rightarrow$ (2.8) $\Rightarrow$ (2.1). To show that (2.1) $\Rightarrow$ (1.25), let $f\in L^2$ be any compactly supported function. Then by using the properties that for every fixed m, $\alpha_{m,\ell}=0$ except for finitely many ℓ, and that both ϕ and f have compact support, it is clear that only finitely many of the values

$$\beta_\ell(f):=\sum_m \alpha_{m,\ell}\langle f,\phi(2\cdot-m)\rangle, \qquad \ell\in\mathbf{Z},$$

are nonzero. Now, since $\hat{\phi}(\omega)$ is a nontrivial entire function, it follows, by taking the Fourier transform of (2.1), that the trigonometric polynomial $\sum_\ell \beta_\ell(f)e^{-i\ell\omega/2}$ is identically zero, so that $\beta_\ell(f)=0$, $\ell\in\mathbf{Z}$, or equivalently,

$$\left\langle f,\sum_m \alpha_{m,\ell}\phi(2\cdot-m)\right\rangle=0, \qquad \ell\in\mathbf{Z}. \tag{2.9}$$

Fix an arbitrary $\ell\in\mathbf{Z}$. Then the series in (2.9) is a finite sum and hence represents a compactly supported function in L^2. By choosing f to be this function, it follows that

$$\sum_m \alpha_{m,\ell}\phi(2\cdot-m)=0,$$

which implies that the trigonometric polynomial $\sum_m \alpha_{m,\ell}e^{-i\omega/2}$ is identically equal to 0, so that $\alpha_{m,\ell}=0$. ■

Proof of Theorem 1. To prove that (1.26) is a necessary condition, we set

$$\mathcal{Q}(z):=\begin{bmatrix} Q_1(z) & Q_1(-z)\\ \vdots & \vdots \\ Q_N(z) & Q_N(-z)\end{bmatrix},$$

and reformulate (1.23) as

$$\begin{bmatrix}\overline{P(z)}\\ \overline{P(-z)}\end{bmatrix}[P(z) \quad P(-z)]+\mathcal{Q}^*(z)\mathcal{Q}(z)=I_2,$$

or equivalently,

$$I_2-\begin{bmatrix}\overline{P(z)}\\ \overline{P(-z)}\end{bmatrix}[P(z) \quad P(-z)]=\mathcal{Q}^*(z)\mathcal{Q}(z),$$

which, for $|z|=1$, is a nonnegative definite Hermitian matrix, so that

$$\det\left(I_2-\begin{bmatrix}\overline{P(z)}\\ \overline{P(-z)}\end{bmatrix}[P(z) \quad P(-z)]\right)\geq 0, \qquad |z|=1;$$

and this gives

$$|P(z)|^2+|P(-z)|^2\le 1,\qquad |z|=1.$$

The proof of the sufficiency of (1.26) is delayed to that of Theorem 2 below. ■

Before giving the proof of Theorem 2, we need to discuss the process of *decorrelation* of the rectangular matrix $\mathcal{R}(z)$, $N\ge 2$. For completeness, we include a brief description of the so-called polyphase decomposition technique ([6, p. 318], i.e., odd–even polynomial decomposition), as follows.

Write $P(z)$ and $Q_j(z)$, $j=1,\dots,N$, in their polyphase forms:

$$\begin{aligned}\sqrt{2}P(z)&=P_1(z^2)+zP_2(z^2);\\ \sqrt{2}Q_j(z)&=Q_{j1}(z^2)+zQ_{j2}(z^2),\qquad j=1,\dots,N,\end{aligned}\tag{2.10}$$

where $P_i(z)$ and $Q_{ji}(z)$, $i=1,2$; $j=1,\dots,N$, are Laurent polynomials. Observe that

$$\mathcal{R}(z)\frac{\sqrt{2}}{2}\begin{bmatrix}1 & z^{-1}\\ 1 & -z^{-1}\end{bmatrix}=\begin{bmatrix}P_1(z^2) & P_2(z^2)\\ Q_{11}(z^2) & Q_{12}(z^2)\\ \vdots & \vdots\\ Q_{N1}(z^2) & Q_{N2}(z^2)\end{bmatrix}.$$

Thus, we see that

$$\begin{aligned}&\begin{bmatrix}P_1(z^2) & P_2(z^2)\\ Q_{11}(z^2) & Q_{12}(z^2)\\ \vdots & \vdots\\ Q_{N1}(z^2) & Q_{N2}(z^2)\end{bmatrix}^*\begin{bmatrix}P_1(z^2) & P_2(z^2)\\ Q_{11}(z^2) & Q_{12}(z^2)\\ \vdots & \vdots\\ Q_{N1}(z^2) & Q_{N2}(z^2)\end{bmatrix}\\ &=\frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ z & -z\end{bmatrix}\mathcal{R}^*(z)\mathcal{R}(z)\frac{\sqrt{2}}{2}\begin{bmatrix}1 & z^{-1}\\ 1 & -z^{-1}\end{bmatrix},\end{aligned}\tag{2.11}$$

and it follows from (1.23), that

$$\begin{bmatrix}P_1(z^2) & P_2(z^2)\\ Q_{11}(z^2) & Q_{12}(z^2)\\ \vdots & \vdots\\ Q_{N1}(z^2) & Q_{N2}(z^2)\end{bmatrix}^*\begin{bmatrix}P_1(z^2) & P_2(z^2)\\ Q_{11}(z^2) & Q_{12}(z^2)\\ \vdots & \vdots\\ Q_{N1}(z^2) & Q_{N2}(z^2)\end{bmatrix}=I_2,\qquad |z|=1.\tag{2.12}$$

It is clear that (2.12) also implies (1.23). To simplify notations, we set $u=z^2$ and observe that the condition (2.12) for the polynomial symbols is satisfied, provided that $(N+1)(N-1)$ Laurent polynomials $P_i(z)$, $Q_{ji}(z)$, where $i=3,\dots,N+1$ and $j=$

$1, \ldots, N$, can be found such that the Laurent polynomial matrix

$$\begin{bmatrix} P_1(u) & \ldots & P_{N+1}(u) \\ Q_{11}(u) & \ldots & Q_{1,N+1}(u) \\ \vdots & \ldots & \vdots \\ Q_{N1}(u) & \ldots & Q_{N,N+1}(u) \end{bmatrix} \tag{2.13}$$

is a unitary matrix on $|u| = 1$. We now turn to the proof of Theorem 2.

Proof of Theorem 2. Let $P_1(z)$ and $P_2(z)$ be the polyphase components of $P(z)$, that is,

$$\sqrt{2}P(z) = P_1(z^2) + zP_2(z^2).$$

Since

$$|P(z)|^2 + |P(-z)|^2 = |P_1(z^2)|^2 + |P_2(z^2)|^2,$$

we have, by (1.26) with $u = z^2$,

$$|P_1(u)|^2 + |P_2(u)|^2 \le 1, \qquad |u| = 1.$$

By the Riesz lemma [6, Lemma 6.1.3], we can find a Laurent polynomial $P_3(u)$ that satisfies

$$|P_1(u)|^2 + |P_2(u)|^2 + |P_3(u)|^2 = 1. \tag{2.14}$$

Next multiply a diagonal matrix, $\operatorname{diag}(u^{t_1}, u^{t_2}, u^{t_3})$ to the left of $[P_1(u), P_2(u), P_3(u)]^*$, where $t_1, t_2, t_3 \in \mathbf{Z}$ are so chosen that each component of

$$\left[u^{t_1}\overline{P_1(u)},\ u^{t_2}\overline{P_2(u)},\ u^{t_3}\overline{P_3(u)}\right]^T = \sum_{j=0}^{n} \mathbf{a}_j u^j \tag{2.15}$$

is a polynomial in u with the lowest degree, where $\mathbf{a}_j \in \mathbf{R}^3$ with $\mathbf{a}_0 \ne 0$ and $\mathbf{a}_n \ne 0$. Now we apply the unitary matrix extension technique in [11]. It follows from (2.14) that

$$\left(\sum_{j=0}^{n} \mathbf{a}_j u^j\right)^* \left(\sum_{j=0}^{n} \mathbf{a}_j u^j\right) = 1, \qquad |u| = 1,$$

and consequently, $\mathbf{a}_0^T \mathbf{a}_n = 0$. We next consider the 3×3 Householder matrix

$$H_1 := I_3 - \frac{2}{|\mathbf{v}|^2}\mathbf{v}\mathbf{v}^T,$$

(see [7, p. 195]), where $\mathbf{v} := \mathbf{a}_n \pm |\mathbf{a}_n|\mathbf{e}_1$ with $\mathbf{e}_1 := (1, 0, 0)^T$ and the $+$ or $-$ signs are so chosen that $\mathbf{v} \ne 0$. Then

$$H_1\mathbf{a}_n = \mp|\mathbf{a}_n|\mathbf{e}_1. \tag{2.16}$$

Indeed, since $|\mathbf{v}|^2 = 2|\mathbf{a}_n|^2 \pm 2|\mathbf{a}_n|\mathbf{e}_1^T\mathbf{a}_n$, and $\mathbf{v}^T\mathbf{e}_1 = \mathbf{a}_n^T\mathbf{e}_1 \pm |\mathbf{a}_n|$, we have

$$H_1\mathbf{a}_n = \left(I_3 - \frac{2}{|\mathbf{v}|^2}\mathbf{v}\mathbf{v}^T\right)(\mathbf{v} \mp |\mathbf{a}_n|\mathbf{e}_1)$$
$$= \mathbf{v} - 2\mathbf{v} \mp |\mathbf{a}_n|\mathbf{e}_1 \pm \frac{\mathbf{a}_n^T\mathbf{e}_1 \pm |\mathbf{a}_n|}{|\mathbf{a}_n| \pm \mathbf{e}_1^T\mathbf{a}_n}\mathbf{v} = \mp|\mathbf{a}_n|\mathbf{e}_1.$$

Also, we note that the symmetric matrix H_1 is orthonormal, since

$$H_1^T H_1 = I_3 - \frac{4}{|\mathbf{v}|^2}\mathbf{v}\mathbf{v}^T + \frac{4}{|\mathbf{v}|^4}\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T = I_3.$$

Hence, $(H_1\mathbf{a}_0)^T(H_1\mathbf{a}_n) = \mathbf{a}_0^T\mathbf{a}_n = 0$, and therefore, by (2.16), the first component of $H_1\mathbf{a}_0$ is 0. Now

$$H_1\left[u^{t_1}\overline{P_1(u)},\ u^{t_2}\overline{P_2(u)},\ u^{t_3}\overline{P_3(u)}\right]^T = \sum_{j=0}^{n}(H_1\mathbf{a}_j)u^j.$$

Therefore, $\operatorname{diag}(u^{-1}, 1, 1)H_1[u^{t_1}\overline{P_1(u)},\ u^{t_2}\overline{P_2(u)},\ u^{t_3}\overline{P_3(u)}]^T$ is also a polynomial vector with unit Euclidean norm on $|u| = 1$ and degree $\le n - 1$. Write

$$\operatorname{diag}(u^{-1}, 1, 1)H_1\left[u^{t_1}\overline{P_1(u)},\ u^{t_2}\overline{P_2(u)},\ u^{t_3}\overline{P_3(u)}\right]^T = \sum_{j=0}^{n_1}\tilde{\mathbf{a}}_j u^j$$

with $n_1 < n$, $\tilde{\mathbf{a}}_0 \neq 0$, $\tilde{\mathbf{a}}_{n_1} \neq 0$. Similarly, define

$$H_2 := I_3 - \frac{2}{|\tilde{\mathbf{v}}|^2}\tilde{\mathbf{v}}\tilde{\mathbf{v}}^T,$$

where $\tilde{\mathbf{v}} := \tilde{\mathbf{a}}_{n_1} \pm |\tilde{\mathbf{a}}_{n_1}|\mathbf{e}_1$ (such that $\tilde{\mathbf{v}} \neq 0$). We repeat this procedure up to $n-1$ times to get a Laurent polynomial matrix

$$H := H_s \operatorname{diag}(u^{-1}, 1, 1)H_{s-1} \cdots \operatorname{diag}(u^{-1}, 1, 1)H_1, \qquad s \le n+1,$$

which is unitary on $|u| = 1$ such that $H[u^{t_1}\overline{P_1(u)},\ u^{t_2}\overline{P_2(u)},\ u^{t_3}\overline{P_3(u)}]^T = \pm\mathbf{e}_1$. Then

$$\left[\overline{P_1(u)},\ \overline{P_2(u)},\ \overline{P_3(u)}\right]^T = \operatorname{diag}(u^{-t_1}, u^{-t_2}, u^{-t_3})H^* \operatorname{diag}(\pm 1, 1, 1)\mathbf{e}_1,$$

or

$$[P_1(u),\ P_2(u),\ P_3(u)]^T = \mathbf{e}_1^T \operatorname{diag}(\pm 1, 1, 1)H \operatorname{diag}(u^{t_1}, u^{t_2}, u^{t_3}).$$

That is, $[P_1(u),\ P_2(u),\ P_3(u)]$ is the first row of the unitary matrix

$$\operatorname{diag}(\pm 1, 1, 1)H \operatorname{diag}(u^{t_1}, u^{t_2}, u^{t_3}), \qquad |u| = 1.$$

Write

$$\operatorname{diag}(\pm 1, 1, 1)H \operatorname{diag}(u^{t_1}, u^{t_2}, u^{t_3}) = \begin{bmatrix} P_1(u) & P_2(u) & P_3(u) \\ Q_{11}(u) & Q_{12}(u) & Q_{13}(u) \\ Q_{21}(u) & Q_{22}(u) & Q_{23}(u) \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} P_1(u) & P_2(u) \\ Q_{11}(u) & Q_{12}(u) \\ Q_{21}(u) & Q_{22}(u) \end{bmatrix}^* \begin{bmatrix} P_1(u) & P_2(u) \\ Q_{11}(u) & Q_{12}(u) \\ Q_{21}(u) & Q_{22}(u) \end{bmatrix} = I_2, \qquad |u| = 1.$$

By setting

$$Q_i(z) := \frac{\sqrt{2}}{2}\left(Q_{i1}(z^2) + zQ_{i2}(z^2)\right), \qquad i = 1, 2,$$

in (1.22) to yield (1.23), we complete the proof of Theorem 2. ∎

Remark 4. We can also choose t_1, t_2, and t_3 such that the right-hand side of (2.15) is a polynomial of u^{-1} with coefficients in $\mathbf{R}^3$. This will be done in Examples 1–3 in Section 3.

Proof of Theorem 3. Consider a Laurent polynomial matrix

$$\mathcal{R}(z) := \begin{bmatrix} P(z) & P(-z) \\ z^{2\ell+1}\overline{P(-z)} & -z^{2\ell+1}\overline{P(z)} \\ Q(z) & Q(-z) \\ z^{2k+1}\overline{Q(-z)} & -z^{2k+1}\overline{Q(z)} \end{bmatrix},$$

for some $k, \ell \in \mathbf{Z}$ and Laurent polynomial $Q(z)$. It is easy to see that

$$\mathcal{R}^*(z)\mathcal{R}(z) = \left(|P(z)|^2 + |P(-z)|^2 + |Q(z)|^2 + |Q(-z)|^2\right) I_2. \tag{2.17}$$

By Lemma 1, we only need to find a symmetric Laurent polynomial $Q(z)$, such that

$$|P(z)|^2 + |P(-z)|^2 + |Q(z)|^2 + |Q(-z)|^2 = 1, \qquad |z| = 1. \tag{2.18}$$

To accomplish this goal, we consider

$$Q(z) = A(z) + z^{4n+1} A\left(\frac{1}{z}\right), \tag{2.19}$$

where

$$A(z) := \sum_{j=0}^{n} a_j z^{2j}, \tag{2.20}$$

with real coefficients a_j and $a_0\, a_n \neq 0$. Hence, $Q(z)$ is symmetric and

$$|Q(z)|^2 + |Q(-z)|^2 = 2\left(|A(z)|^2 + |A(-z)|^2\right) = 4|A(z)|^2, \qquad |z| = 1. \tag{2.21}$$

Since $P(z)$ satisfies (1.26), $1 - |P(z)|^2 - |P(-z)|^2$ is a nonnegative symmetric Laurent polynomial of z^2 for $|z| = 1$. By the Riesz lemma, $A(z)$ in the form of (2.20), which satisfies

$$4|A(z)|^2 = 1 - |P(z)|^2 - |P(-z)|^2,$$

exists. Hence, $Q(z)$ as given by (2.19) is a symmetric polynomial and satisfies (2.18). Consequently, $\psi^1, \psi^2, \psi^3 \in V_1$, with two-scale symbols $z^{2\ell+1}\overline{P(-z)}$, $Q(z)$, and $z^{2k+1}\overline{Q(-z)}$, respectively, are compactly supported symmetric or anti-symmetric wavelets, and $\Psi = \{\psi^1, \psi^2, \psi^3\}$ is a minimum-energy frame associated with ϕ. ■

Remark 5. It is easy to see (and will be elaborated in Section 3) that for cardinal B-splines N_m and interpolating scaling functions ϕ_m^I of arbitrary orders m, there always exist compactly supported symmetric or anti-symmetric $\Psi = \{\psi^1, \psi^2, \psi^3\}$ that are minimum-energy frames associated with N_m or ϕ_m^I, respectively.

Remark 6. In practice, we can find $Q(z)$ in a form slightly different from (2.19), such that $\Psi = \{\psi^1, \psi^2, \psi^3\}$ has smaller supports. This can be seen from the examples in the next section.

3. EXAMPLES OF COMPACTLY SUPPORTED MINIMUM-ENERGY FRAMES

In this section, we give examples of two classes of minimum-energy frames, one associated with the cardinal B-splines N_m in (1.16) and the other associated with the compactly supported interpolating scaling functions ϕ_m^I obtained by taking the autocorrelations of the mth order Daubechies o.n. scaling functions.

3.1. Minimum-energy Frames with Two Generators

3.1.1. Cardinal B-splines. It is well known that the mth order cardinal B-spline N_m has the two-scale relation

$$\hat{N}_m(\omega) = P_m(z)\hat{N}_m\left(\frac{\omega}{2}\right),$$

where $z := e^{-i\omega/2}$ and $P_m(z) = 2^{-m}(1+z)^m$. Observe that $|P_m(z)| = |\cos\theta|^m$ and $|P_m(-z)| = |\sin\theta|^m$, where $\theta = \omega/4$. Hence,

$$|P_m(z)|^2 + |P_m(-z)|^2 = (\cos^2\theta)^m + (\sin^2\theta)^m \le \cos^2\theta + \sin^2\theta = 1.$$

By Theorem 2, there exists a compactly supported minimum-energy frame $\{\psi_m^1, \psi_m^2\}$ associated with N_m.

EXAMPLE 1 (Linear B-splines). For the symbol $P_2(z)$, it is easy to find

$$Q_1(z) := -\frac{1}{4} + \frac{1}{2}z - \frac{1}{4}z^2 \qquad \text{and} \qquad Q_2(z) := \frac{\sqrt{2}}{4}(1-z^2).$$

Hence, ψ_2^1 is symmetric and ψ_2^2 is anti-symmetric (see Fig. 1). This result was already given in [12].

EXAMPLE 2 (Quadratic B-splines). The symbol $P_3(z)$ has polyphase components

$$P^1(u) := \frac{1}{8}(1+3u) \qquad \text{and} \qquad P^2(u) := \frac{1}{8}(3+u).$$

Since

$$\left|\frac{\sqrt{2}}{8}(1+3u)\right|^2 + \left|\frac{\sqrt{2}}{8}(3+u)\right|^2 + \left|\frac{3}{16}(1-u)\right|^2 = 1,$$

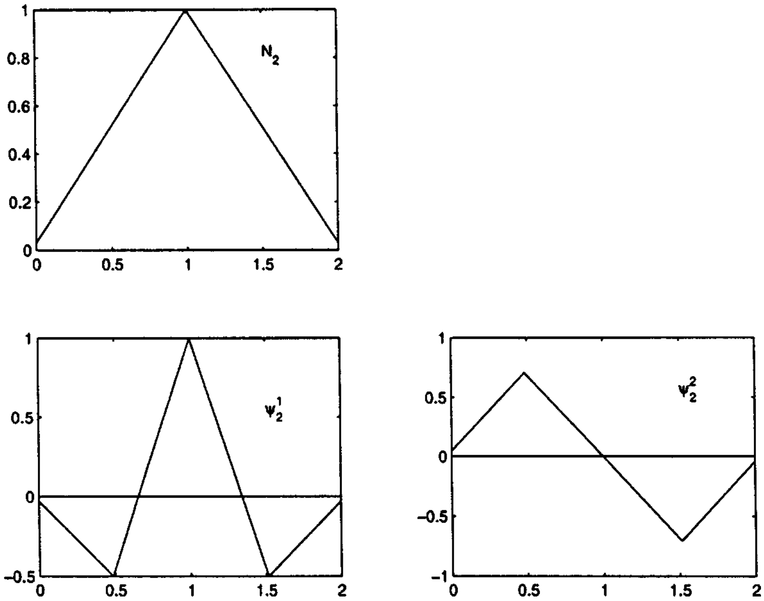

FIG. 1. An MRA tight frame associated with the linear B-spline.

we may set

$$P^3(u) := \frac{3}{16}(1-u). \tag{3.1}$$

Following the proof of Theorem 2, we consider the vector-valued polynomial expression in (2.14) with $\mathbf{a}_n \neq 0$, and attempt to transform $\mathbf{a}_n$ into a constant multiple of the coordinate unit vector $\mathbf{e}_1$ as in (2.15). Instead of using the Householder matrix H_1 as in the proof of Theorem 2, we could have used 2-dimensional unitary matrix rotations. For example, we can first annihilate the last (or third) component of $\mathbf{a}_n$ by rotating the 2-dimensional vector formulated by the second and third components of $\mathbf{a}_n$ and then the second entry of $\mathbf{a}_n$ by rotating the resulting 2-dimensional vector formulated by the first and second components. In this example, we have the unitary matrix extension

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{8}(1+3u) & \frac{\sqrt{2}}{8}(3+u) & \frac{\sqrt{3}}{4}(1-u) \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & -\frac{1}{2} \\ \frac{\sqrt{2}}{8}(1-3u) & \frac{\sqrt{2}}{8}(3-u) & \frac{\sqrt{3}}{4}(1+u) \end{bmatrix}.$$

Hence,

$$Q_1(z) = -\frac{\sqrt{3}}{4}(1-z), \qquad Q_2(z) = \frac{1}{8}(1+3z-3z^2-z^3), \tag{3.2}$$

and both ψ_3^1 and ψ_3^2 are anti-symmetric (see Fig. 2).

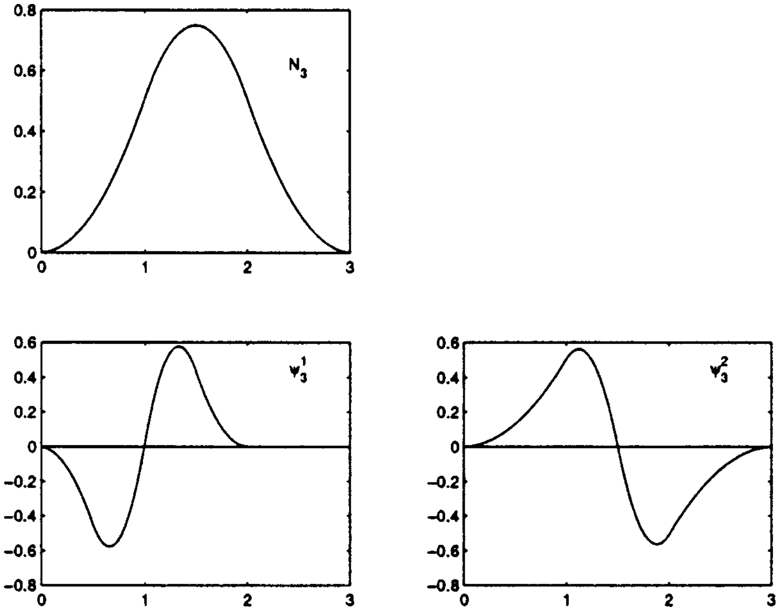

FIG. 2. An MRA tight frame associated with the quadratic B-spline.

EXAMPLE 3 (Cubic B-splines). For the symbol $P_4(z)$, the polyphase components are

$$P^1(u) = \frac{1}{16}(1+6u+u^2) \qquad \text{and} \qquad P^2(u) = \frac{1}{4}(1+u).$$

Now we solve the equation

$$\left|\frac{\sqrt{2}}{16}(1+6u+u^2)\right|^2 + \left|\frac{\sqrt{2}}{4}(1+u)\right|^2 + \left|P^3(u)\right|^2 = 1$$

for $P^3(u)$. By applying the Riesz lemma, one of the solutions is given by

$$P^3(u) = \frac{1}{4} + \frac{\sqrt{14}}{16} - \frac{\sqrt{14}}{8}u - \left(\frac{1}{4} - \frac{\sqrt{14}}{16}\right)u^2. \tag{3.3}$$

Again, applying three 2-dimensional vector rotations as in Example 2 and one Householder transform as in the proof of Theorem 2, we can compute the unitary matrix extension as follows.

$$\begin{bmatrix} 1/2 & \sqrt{2}b & -\sqrt{2}ru \\ \sqrt{2}b & 1-4b^2 & 4bru \\ -\sqrt{2}r & 4br & (1-4r^2)u \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{c}{r} & -\frac{a}{r} & 0 \\ \frac{a}{r} & \frac{c}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2bu & 0 & -2au \\ 0 & 1 & 0 \\ 2a & 0 & 2b \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\sqrt{2}}{16}(1+6u+u^2) & \frac{\sqrt{2}}{4}(1+u) & \frac{1}{4} + \frac{\sqrt{14}}{16} - \frac{\sqrt{14}}{8}u - \left(\frac{1}{4} - \frac{\sqrt{14}}{16}\right)u^2 \\ 4\sqrt{2}ar^2 + \frac{1-2r^2}{4r}u - \frac{b}{4}u^2 & r - \frac{1}{16r} - bu & 4\sqrt{2}br^2 + \frac{13\sqrt{2}}{128r}u + \frac{a}{4}u^2 \\ \frac{r}{4} - \frac{b}{2}u - \frac{b^2}{r}u^2 & \sqrt{2}a + b - \frac{b^2}{r}u & 4\sqrt{2}b^2r + \left(\frac{a}{2} + \frac{1}{16a}\right)u + \frac{\sqrt{2}}{128r}u^2 \end{bmatrix},$$

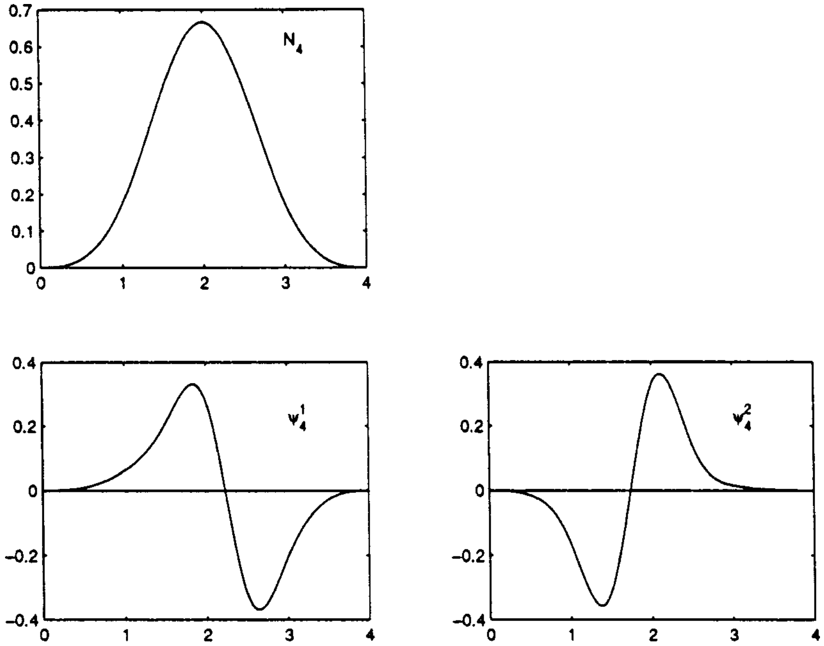

FIG. 3. An MRA tight frame for cubic B-spline close to anti-symmetry.

where

$$a := \frac{\sqrt{8-2\sqrt{14}}}{8}, \qquad b := \frac{\sqrt{8+2\sqrt{14}}}{8},$$
$$c := \frac{\sqrt{2}}{4}, \qquad r := \sqrt{a^2+c^2} = \frac{\sqrt{16+2\sqrt{14}}}{8}. \tag{3.4}$$

Hence, we have

$$Q_1(z) = 4ar^2 + \left(\frac{\sqrt{2}r}{2} - \frac{\sqrt{2}}{32r}\right)z + \frac{\sqrt{2}-2\sqrt{2}r^2}{8r}z^2 - \frac{\sqrt{2}b}{2}z^3 - \frac{\sqrt{2}b}{8}z^4;$$
$$Q_2(z) = \frac{\sqrt{2}r}{8} + \left(a + \frac{\sqrt{2}b}{2}\right)z - \frac{\sqrt{2}b}{4}z^2 - \frac{\sqrt{2}b}{2r}z^3 - \frac{\sqrt{2}b^2}{2r}z^4.$$

From this, we extract a one-parameter family solution as follows:

$$\begin{aligned} Q_1^\theta(z) &= \cos\theta\, Q_1(z) + \sin\theta\, Q_2(z); \\ Q_2^\theta(z) &= -\sin\theta\, Q_1(z) + \cos\theta\, Q_2(z), \qquad \theta \in [0, 2\pi]. \end{aligned} \tag{3.5}$$

The choice of $\theta = 0.5$ gives an almost symmetric solution (see Fig. 3).

3.1.2. Compactly supported interpolating scaling functions. Let $\phi_m^I(x)$ be the compactly supported interpolating scaling function with two-scale symbol

$$P_m^I(z) = z^{-m}\left(\frac{1+z}{2}\right)^{2m} \sum_{k=0}^{m-1} 2^{-2k} \binom{m+k-1}{k} (2-z-z^{-1})^k, \tag{3.6}$$

which satisfies

$$P_m^I(z) \geq 0 \qquad \text{and} \qquad P_m^I(z) + P_m^I(-z) = 1, \qquad |z| = 1. \tag{3.7}$$

Then we have, for $|z| = 1$,

$$|P_m^I(z)|^2 + |P_m^I(-z)|^2 \leq |P_m^I(z)| + |P_m^I(-z)| = P_m^I(z) + P_m^I(-z) = 1.$$

By Theorem 2, we can find a compactly supported minimum-energy frame $\Psi_m^I = \{\psi_m^1, \psi_m^2\}$ associated with ϕ_m^I. For this class of examples, we can even give an explicit formulation.

Let $P_{m,e}^I(z)$ and $P_{m,o}^I(z)$ be the polyphase components (as in (2.10)) of $P_m^I(z)$. By (3.7), we have

$$P_{m,e}^I(z) = \frac{\sqrt{2}}{2}. \tag{3.8}$$

Based on the construction procedure in the proof of Theorem 2, we can find a Laurent polynomial P_3, such that

$$|P_{m,e}^I(u)|^2 + |P_{m,o}^I(u)|^2 + |P_3(u)|^2 = I, \qquad u = z^2. \tag{3.9}$$

Actually, $P_3(u)$ satisfies

$$\begin{aligned} |P_3(u)|^2 &= 1 - (|P_{m,e}^I(u)|^2 + |P_{m,o}^I(u)|^2) \\ &= (P_m^I(z) + P_m^I(-z))^2 - (|P_m^I(z)|^2 + |P_m^I(-z)|^2) \\ &= 2P_m^I(z)P_m^I(-z), \end{aligned}$$

and from this, we can deduce that

$$P_3(u) = \sqrt{2}P_m^D(z)P_m^D(-z), \tag{3.10}$$

where $P_m^D(z)$ is the symbol of the Daubechies scaling function ϕ_m^D.

By (3.8) and (3.9), we get

$$|P_{m,o}^I(u)|^2 + |P_3(u)|^2 = \frac{1}{2}. \tag{3.11}$$

It follows that

$$\begin{bmatrix} \sqrt{2}P_{m,o}^I(u) & \sqrt{2}P_3(u) \\ \sqrt{2}P_3(1/u) & -\sqrt{2}P_{m,o}^I(1/u) \end{bmatrix}$$

is a unitary matrix for $|u| = 1$, and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}P_{m,o}^I(u) & \sqrt{2}P_3(u) \\ 0 & \sqrt{2}P_3(1/u) & -\sqrt{2}P_{m,o}^I(1/u) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ P_{m,o}^I(1/u) \\ P_3(1/u) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}. \tag{3.12}$$

By multiplying both sides of (3.12) by

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to the right, we get

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & P^I_{m,o}(u) & P_3(u) \\ \frac{\sqrt{2}}{2} & -P^I_{N,o}(u) & -P_3(u) \\ 0 & \sqrt{2}P_3(1/u) & -\sqrt{2}P^I_{m,o}(1/u) \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ P^I_{m,o}(1/u) \\ P_3(1/u) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{3.13}$$

Hence, we can write out the symbols for ψ^1_m and ψ^2_m, namely,

$$\begin{aligned} Q^1_m(z) &= \frac{1}{2} - \frac{\sqrt{2}}{2} z P^I_{m,o}(z^2) = 1 - P^I_m(z); \\ Q^2_m(z) &= z P_3(1/z^2) = \sqrt{2} P^D_m\left(\frac{1}{z}\right) P^D_m\left(-\frac{1}{z}\right). \end{aligned} \tag{3.14}$$

Note that ψ^1_m with two-scale symbol $Q^1_m(z)$ is symmetric, but ψ^2_m is not, due to the asymmetry of $P^D_m(z)$ that governs the Daubechies scaling function.

EXAMPLE 4. Construction of Ψ^I_2:

$$\begin{aligned} P^I_2(z) &= -\frac{1}{32z^3} + \frac{9}{32z} + \frac{1}{2} + \frac{9z}{32} - \frac{z^3}{32}, \\ Q^1_2(z) &= +\frac{1}{32z^3} - \frac{9}{32z} + \frac{1}{2} - \frac{9z}{32} + \frac{z^3}{32}, \\ Q^2_2(z) &= -\left(\frac{\sqrt{6}+2\sqrt{2}}{32}\right)\frac{1}{z^3} + \left(\frac{\sqrt{6}+6\sqrt{2}}{32}\right)\frac{1}{z} \\ &\quad + \left(\frac{\sqrt{6}-6\sqrt{2}}{32}\right)z + \left(\frac{2\sqrt{2}-\sqrt{6}}{32}\right)z^3 \end{aligned} \tag{3.15}$$

(see Fig. 4).

EXAMPLE 5. Construction of Ψ^I_3:

$$\begin{aligned} P^I_3(z) &= \frac{3}{512z^5} - \frac{25}{512z^3} + \frac{75}{256z} + \frac{1}{2} + \frac{75z}{256} - \frac{25z^3}{512} + \frac{3z^5}{512}, \\ Q^1_3(z) &= -\frac{3}{512z^5} + \frac{25}{512z^3} - \frac{75}{256z} + \frac{1}{2} - \frac{75z}{256} + \frac{25z^3}{512} - \frac{3z^5}{512}, \\ Q^2_3(z) &= \frac{\sqrt{2}}{8}\left(\frac{z-1/z}{2}\right)^3 (az^{-2} + b + cz^2), \end{aligned} \tag{3.16}$$

where

$$a = 1 + \frac{\sqrt{10}}{4} + \frac{1}{8}\sqrt{95+32\sqrt{10}}, \qquad b = -2 + \frac{\sqrt{10}}{2},$$

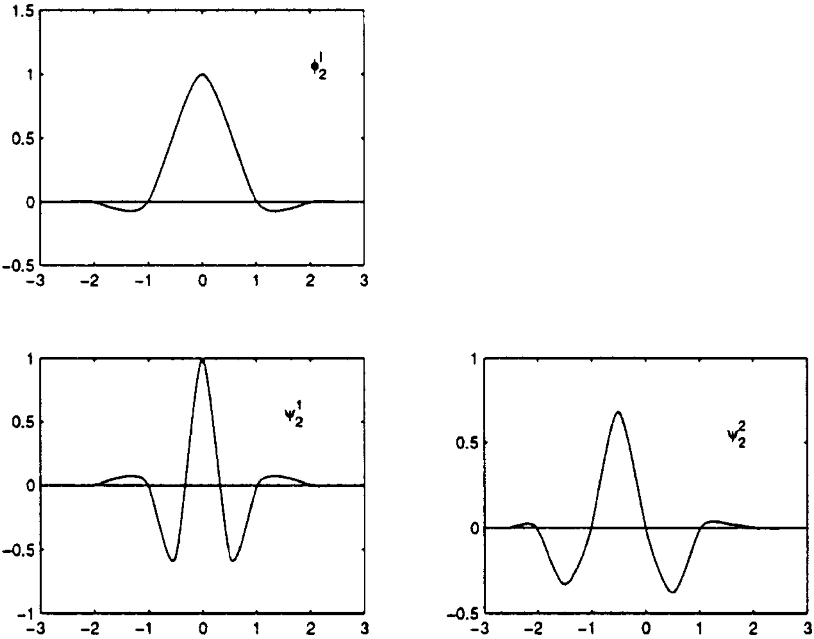

FIG. 4. An MRA tight frame associated with the interpolating scaling function ϕ_2^I.

$$c = 1 + \frac{\sqrt{10}}{4} - \frac{1}{8}\sqrt{95 + 32\sqrt{10}}$$

(see Fig. 5).

3.2. Symmetric Tight Frames with Three Generators

Based on the constructive proof of Theorem 3, we give examples of compactly supported symmetric and/or anti-symmetric minimum-energy (tight) frames $\Psi = \{\psi^1, \psi^2, \psi^3\}$ associated with the cardinal B-splines N_4, N_5, and N_6 the interpolating scaling functions ϕ_2^I and ϕ_3^I.

EXAMPLE 6. Symmetric tight frame associated with the cubic B-spline N_4:

$$\begin{aligned} P(z) &= \left(\frac{1+z}{2}\right)^4, & Q_1(z) &= zP(-z), \\ Q_2(z) &= \frac{1}{16}(1-z^2)(1-2\sqrt{7}z+z^2), & Q_3(z) &= zQ_2(-z) \end{aligned} \tag{3.17}$$

(see Fig. 6).

EXAMPLE 7. Symmetric tight frame associated with the quartic B-spline N_5:

$$\begin{aligned} P(z) &= \left(\frac{1+z}{2}\right)^5, & Q_1(z) &= P(-z), \\ Q_2(z) &= \frac{10}{32}(1-z^2)(1-2\sqrt{3}z+z^2), & Q_3(z) &= zQ_2(-z). \end{aligned} \tag{3.18}$$

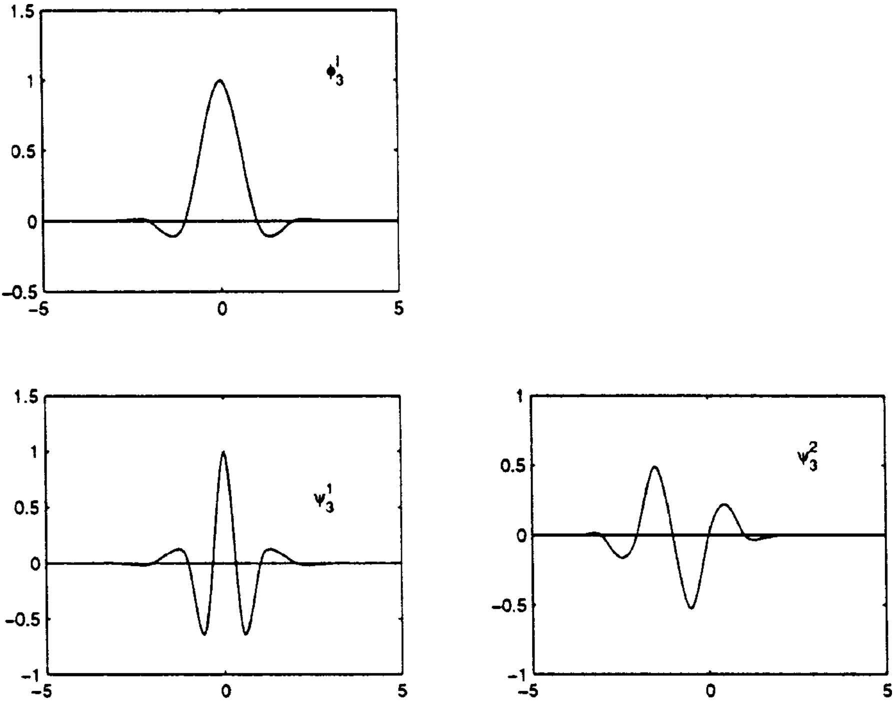

FIG. 5. An MRA tight frame associated with the interpolating scaling function ϕ_3^I.

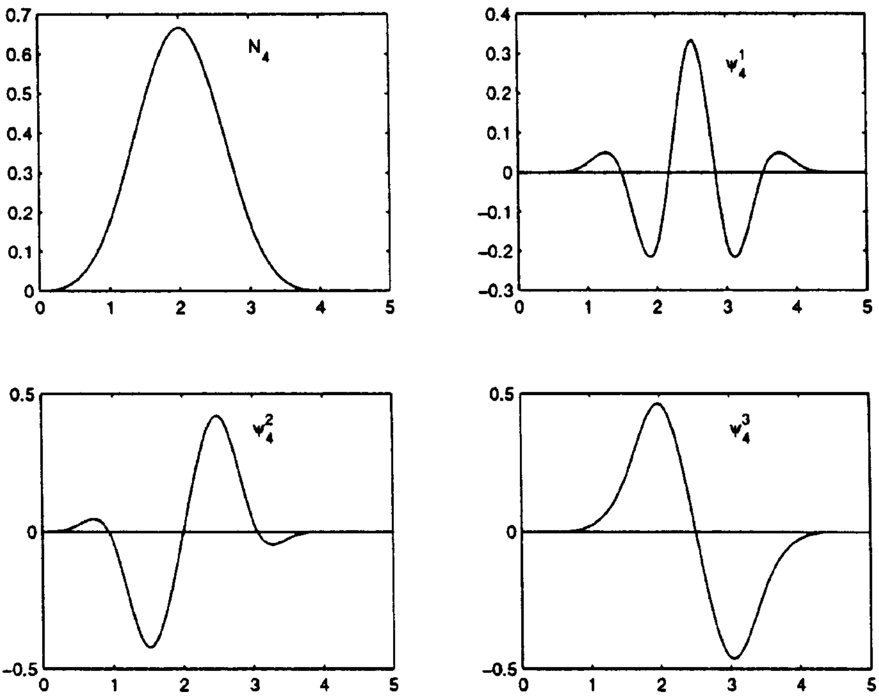

FIG. 6. Anti-symmetric tight frame associated with the cubic B-spline N_4.

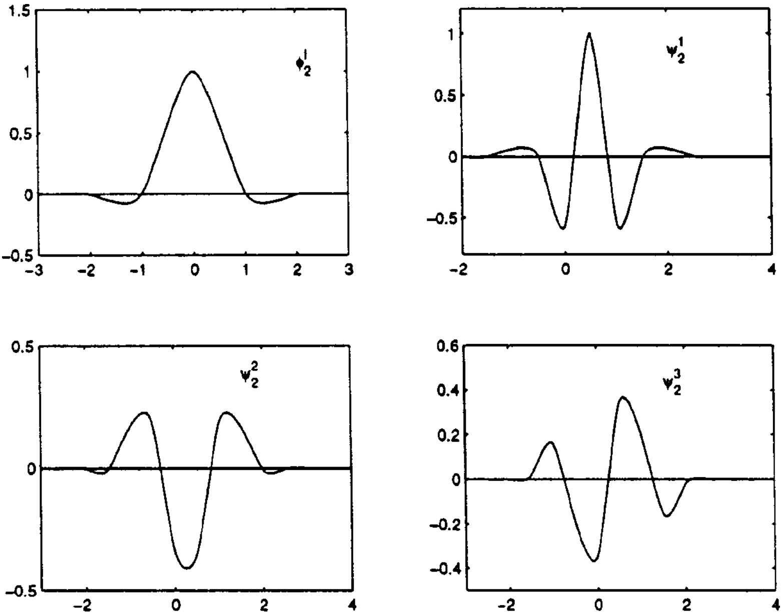

FIG. 7. A symmetric tight frame associated with the interpolating scaling function ϕ_2^I.

EXAMPLE 8. Symmetric tight frame associated with the quintic B-spline N_6:

$$P(z) = \left(\frac{1+z}{2}\right)^6, \qquad Q_1(z) = zP(-z),$$

$$Q_2(z) = \frac{1}{4}z^2(1-z^2)\left(\frac{\sqrt{31}}{4} + \frac{1}{8} + \frac{\sqrt{16-2\sqrt{31}}}{8}(z+z^{-1}) + \frac{1}{16}(z^2+z^{-2})\right), \quad (3.19)$$

$$Q_3(z) = zQ_2(-z).$$

EXAMPLE 9. Symmetric tight frame associated with the interpolating scaling function ϕ_2^I:

$$P_2^I(z) = z^{-2}\left(\frac{1+z}{2}\right)^4\left(2 - \frac{1}{2}(z+z^{-1})\right), \qquad Q_1(z) = zP_2^I(-z),$$

$$Q_2(z) = \frac{\sqrt{2}}{32}z^{-2}(1+z)^3(1-z)^2\left(2-(1-\sqrt{3}/2)(z+z^{-1})\right), \qquad Q_3(z) = Q_2(-z) \quad (3.20)$$

(see Fig. 7).

EXAMPLE 10. Symmetric tight frame associated with the interpolating scaling function ϕ_3^I:

$$P_3^I(z) = z^{-3}\left(\frac{1+z}{2}\right)^6\left(\frac{19}{4} - \frac{9}{4}(z+z^{-1}) + \frac{3}{8}(z^2+z^{-2})\right), \qquad Q_1(z) = zP_3^I(-z),$$

$$Q_2(z) = z^{-3}\left(\frac{1-z^2}{4}\right)^3\left(\frac{13}{4} + \frac{\sqrt{15}}{4}(z+z^{-1}) - \frac{3}{8}(z^2+z^{-2})\right), \qquad Q_3(z) = zQ_2(-z) \quad (3.21)$$

(see Fig. 8).

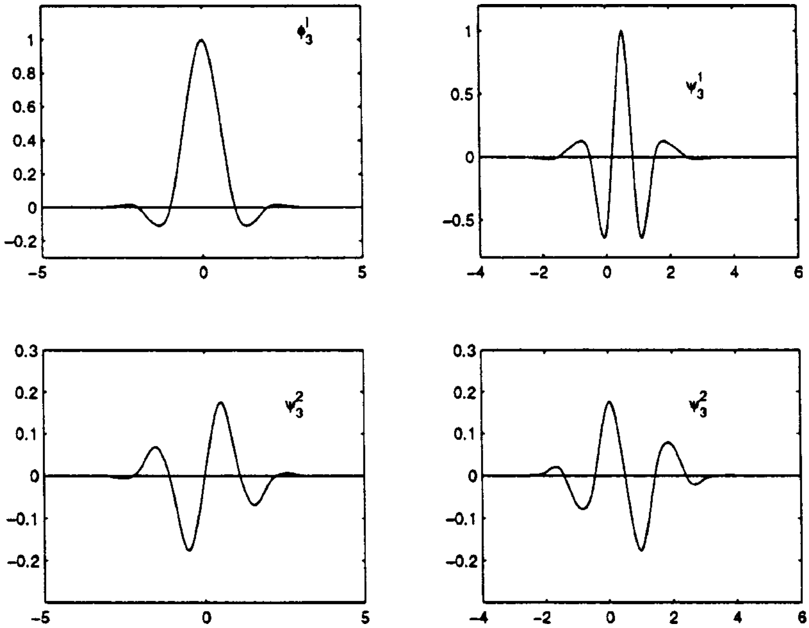

FIG. 8. A symmetric tight frame associated with the interpolating scaling function ϕ_3^I.

4. MINIMUM-ENERGY FRAME DECOMPOSITION

Suppose that a refinable function ϕ with two-scale symbol $P(z) \in \mathcal{W}$ has an associated minimum-energy frame $\Psi = \{\psi^1, \dots, \psi^N\}$ with two-scale symbols $Q_1(z), \dots, Q_N(z) \in \mathcal{W}$. Then by Lemma 1, the $(N+1) \times 2$ matrix $\mathcal{R}(z)$ in (1.22) formulated by these symbols satisfies (1.23). In the proof of this lemma, we have the decomposition relation (2.7). Hence, by setting

$$U_j := \text{clos}_{L^2} \langle \psi^i_{j,k} : i = 1, \dots, N;\ k \in \mathbf{Z} \rangle, \tag{4.1}$$

it follows that

$$V_{j+1} = V_j + U_j, \qquad j \in \mathbf{Z}, \tag{4.2}$$

but this is not a direct sum decomposition, because

$$V_j \cap U_j \neq \{0\}.$$

Indeed, let $\eta(x) \in V_0 \cap U_0$ and write

$$\eta(x) = \sum_{k \in \mathbf{Z}} s_k \phi(x - k) = \sum_{i=1}^{N} \sum_{k \in \mathbf{Z}} t^i_k \psi^i(x - k), \tag{4.3}$$

or equivalently, $\hat{\eta}(\omega) = S(z^2)\hat{\phi}(\omega) = \sum_{i=1}^N T^i(z^2)\hat{\psi}^i(\omega)$, with $z = e^{i\omega/2}$, and

$$S(z) = \sum_{k \in \mathbf{Z}} s_k z^k \qquad \text{and} \qquad T^i(z) = \sum_{k \in \mathbf{Z}} t^i_k z^k, \qquad i = 1, \dots, N.$$

By the two-scale relations (1.18) and (1.20), we have

$$S(z^2)P(z) - \sum_{i=1}^{N} T^i(z^2)Q^i(z) = 0.$$

Let $P_e(z)$, $P_o(z)$ be the polyphase components of $P(z)$, and $Q_e^i(z)$, $Q_o^i(z)$ be the polyphase components of $Q^i(z)$. Then, we have

$$S(z^2)P_e(z^2) - \sum_{i=1}^{N} T^i(z^2)Q_e^i(z^2) = 0,$$

$$S(z^2)P_o(z^2) - \sum_{i=1}^{N} T^i(z^2)Q_o^i(z^2) = 0.$$

That is,

$$\left[S(z^2), -T^1(z^2), \ldots, -T^N(z^2)\right] \begin{bmatrix} P_e(z^2) & P_o(z^2) \\ Q_e^1(z^2) & Q_o^1(z^2) \\ \vdots & \vdots \\ Q_e^N(z^2) & Q_o^N(z^2) \end{bmatrix} = 0. \tag{4.4}$$

For $N \geq 2$, there exist (non-trivial) Laurent polynomials $S(z)$ and $T^i(z)$, $i = 1, \ldots, N$, which satisfy (4.4) so that $0 \not\equiv \eta(x) \in V_0 \cap U_0$ exists (see the proof of Theorem 2).

So, what is the significance of the decomposition formulation (1.13) which, in the first place, is equivalent to the definition of minimum-energy frames associated with ϕ?

To answer this question, let us first consider the projection operators P_j of L^2 onto the nested subspaces V_j defined by

$$P_j f := \sum_{k \in \mathbf{Z}} \langle f, \phi_{j,k} \rangle \phi_{j,k}. \tag{4.5}$$

The decomposition formula (1.13) can then be written as

$$P_{j+1} f - P_j f = \sum_{i=1}^{N} \sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k}^i \rangle \psi_{j,k}^i.$$

In other words, the error term $g_j := P_{j+1} f - P_j f$ between consecutive projections is given by the frame expansion

$$g_j = \sum_{i=1}^{N} \sum_{k \in \mathbf{Z}} \langle f, \psi_{j,k}^i \rangle \psi_{j,k}^i. \tag{4.6}$$

The importance of this frame expansion as compared to any other expansion

$$g_j = \sum_{i=1}^{N} \sum_{k \in \mathbf{Z}} c_{i,k} \psi_{j,k}^i \tag{4.7}$$

of the same g_j is that the energy in (4.6) is minimum in the sense that

$$\sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|\langle f,\psi^i_{j,k}\rangle|^2 \le \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|c_{i,k}|^2. \tag{4.8}$$

Indeed, by using both (4.6) and (4.7), we have

$$\langle g_j, f\rangle = \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|\langle f,\psi^i_{j,k}\rangle|^2 = \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}} c_{i,k}\overline{\langle f,\psi^i_{j,k}\rangle}$$

(and hence the last quantity is real), so that

$$\begin{aligned} 0 &\le \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|c_{i,k}-\langle f,\psi^i_{j,k}\rangle|^2 \\ &= \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|c_{i,k}|^2 - 2\sum_{i=1}^{N}\sum_{k\in\mathbf{Z}} c_{i,k}\overline{\langle f,\psi^i_{j,k}\rangle} + \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|\langle f,\psi^i_{j,k}\rangle|^2 \\ &= \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|c_{i,k}|^2 - \sum_{i=1}^{N}\sum_{k\in\mathbf{Z}}|\langle f,\psi^i_{j,k}\rangle|^2, \end{aligned}$$

from which (4.8) follows.

We next discuss minimum-energy (wavelet) frame decomposition and reconstruction. Suppose we have a minimum-energy frame $\Psi = \{\psi^1,\dots,\psi^N\}$ associated with a refinable function ϕ. For an $f\in L^2$, consider

$$c_{j,k} = \langle f,\phi_{j,k}\rangle; \qquad d^i_{j,k} = \langle f,\psi^i_{j,k}\rangle, \qquad i=1,\dots,N. \tag{4.9}$$

Then we can derive the decomposition and reconstruction formulas that are similar to those of orthonormal wavelets.

1° Decomposition algorithm. Suppose the coefficients $\{c_{j+1,\ell} : \ell\in\mathbf{Z}\}$ are known. By the two-scale relations (1.18) and (1.20), we have

$$\begin{aligned} \phi_{j,\ell}(x) &= \frac{1}{\sqrt{2}}\sum_k p_{k-2\ell}\phi_{j+1,k}(x); \\ \psi^i_{j,\ell}(x) &= \frac{1}{\sqrt{2}}\sum_k q^i_{j,k-2\ell}\phi_{j+1,k}(x), \qquad i=1,\dots,N. \end{aligned} \tag{4.10}$$

Hence, the decomposition algorithm is given by

$$\begin{aligned} c_{j,\ell} &= \frac{1}{\sqrt{2}}\sum_k p_{k-2\ell}c_{j+1,k}; \\ d^i_{j,\ell} &= \frac{1}{\sqrt{2}}\sum_k q^i_{k-2\ell}c_{j+1,k}, \qquad i=1,\dots,N;\; j\in\mathbf{Z}. \end{aligned} \tag{4.11}$$

2° Reconstruction algorithm. From (2.7), it follows that

$$\phi_{j+1,\ell}(x) = \frac{1}{\sqrt{2}} \sum_k \left\{ p_{\ell-2k}\phi_{j,k}(x) + \sum_{i=1}^{N} q^i_{\ell-2k}\psi^i_{j,k}(x-k) \right\}, \qquad \ell \in \mathbf{Z}. \tag{4.12}$$

Taking the inner products on both sides of (4.12) with f, we have

$$c_{j+1,\ell} = \frac{1}{\sqrt{2}} \sum_k \left\{ p_{\ell-2k}c_{j,k} + \sum_{i=1}^{N} q^i_{\ell-2k}d^i_{j,k} \right\}. \tag{4.13}$$

By using statement (iii) in Lemma 1, we see that $\{c_{j+1,\ell},\ \ell \in \mathbf{Z}\}$ in (4.13) is the same as $\{c_{j+1,k},\ k \in \mathbf{Z}\}$ in (4.11).

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REFERENCES

1. C. K. Chui, “An Introduction to Wavelets,” Academic Press, Boston, 1992.
2. C. K. Chui, “Wavelets: A Mathematical Tool for Signal Analysis,” SIAM, Philadelphia, 1997.
3. C. K. Chui and J. Z. Wang, A study of asymptotically optimal time-frequency localization by scaling functions and wavelets, *Ann. Numer. Math.* **4** (1997), 193–216.
4. A. Cohen, I. Daubechies, and J. C. Feauveau, Biorthogonal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **45** (1992), 485–460.
5. I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.
6. I. Daubechies, “Ten Lectures on Wavelets,” CBMS-NSF Series in Applied Math., Vol. 61, SIAM, Philadelphia, 1992.
7. G. H. Golub and C. F. Van Loan, “Matrix Computations,” Johns Hopkins Univ. Press, Baltimore, 1989.
8. B. Han, On dual wavelet tight frames, *Appl. Comput. Harmonic Anal.* **4** (1997), 380–413.
9. E. Hernández and G. Weiss, “A First Course on Wavelets,” CRC Press, Boca Raton, FL, 1996.
10. W. Lawton, Tight frames of compactly supported affine wavelets, *J. Math. Phys.* **31** (1990), 1898–1901.
11. W. Lawton, S. L. Lee, and Z. Shen, An algorithm for matrix extension and wavelet construction, *Math. Comp.* **65** (1996), 723–737.
12. A. Ron and Z. Shen, Affine systems in $L_2(R^d)$: the analysis of the analysis operator, *J. Functional Anal. Appl.* **148** (1997), 408–447.
13. A. Ron and Z. Shen, Compactly supported tight affine spline frames in $L_2(R^d)$, *Math. Comp.* **67** (1998), 191–207.
14. A. Ron and Z. Shen, Construction of compactly supported affine frames in $L_2(R^d)$, *in* “Advances in Wavelets” (K. S. Lau, Ed.), Springer-Verlag, New York, 1998.